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Orthogonality and Proportional Norms

By

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Abstract

Two norms on a real vectorspace define the same orthogonality relation iff they are proportional. The aim of this note is to give a proof of this statement with a minimum of results on convex sets, convex functions and real analysis. Needed is only the right derivative of a convex function and the théorème des accroissements finis as it is called by H. Cartan.

G. D. Birkhoff, R. C. James and others (see [1]) used in several important papers the following concept of orthogonality in real normed linear spaces $(X, \| \ \|)$.

Definition 1. *Let* $x, y \in X$. *We say* $x \perp y$ *(*x *is orthogonal to* y*) iff* $\|x + \lambda y\| \geq \|x\|$ *for all* $\lambda \in \mathbb{R}$.

Let us now think of two norms $\| \ \|_1, \| \ \|_2$ on X, then we can ask, when they determine the same orthogonality relation on X. If one analyzes the geometrical meaning of this orthogonality relation, then it seems that the following has to be true.

Theorem. *Two norms* $\| \ \|_1, \| \ \|_2$ *on* X *determine the same orthogonality relation (i.e.* $\perp_1 = \perp_2$*) iff they are proportional (i.e. There exists a number* $\sigma \in \mathbb{R}_{>0}$ *with* $\|x\|_1 = \sigma \|x\|_2$ *for all* $x \in X$.)

This theorem for example is useful in the proof of Theorem 4.17 in [2] as Prof. R. Ger pointed out. Every proof of this theorem will use several basic facts on convex sets and convex functions. Our aim is to give a proof relying on a minimum of these facts and therefore we will use only, that a convex function has a right derivative at every inner point of its domain of definition. For this purpose let us fix some notations.

Definition 2. *Let $(X, \| \|)$ be a real normed linear space and $x, y \in X$ linearly independent. With x, y we always can define the following functions*

$$g(x, y; \bullet): \mathbb{R} \to \mathbb{R}, \quad g(x, y; \mu) := \| x + \mu y \| \quad \text{for all } \mu \in \mathbb{R},$$

$$c(x, y; \bullet): \mathbb{R} \to X, \quad c(x, y; \mu) := \frac{1}{g(x, y; \mu)} (x + \mu y) \quad \text{for all } \mu \in \mathbb{R}.$$

$g^+(x, y; \bullet)$, $c^+(x, y; \bullet)$ *are the right derivatives of these functions.*

Lemma 1. *Let $(X, \| \|)$ be any real normed linear space, $x, y \in X$ linearly independent, then the following statements are true.*

1. $c^+(x, y; 0) = -g^+(x, y; 0) \dfrac{x}{\|x\|^2} + \dfrac{y}{\|x\|}$
2. $c^+(x, y; 0)$ *and* $c(x, y; 0)$ *are linearly independent.*
3. *If* $x \perp y$, *then* $-g^+(x, y; 0) \leq 0$.
4. *If* $\tilde{y} := \eta(y + \kappa x)$ *with* $\kappa, \eta \in \mathbb{R}$, $\eta > 0$, *then* $c^+(x, \tilde{y}; 0) = \eta c^+(x, y; 0)$
5. $c(x, y - x; \lambda + \mu) = c(x + \lambda(y - x), y - x; \mu)$ *and* $c^+(x, y - x; \lambda) = c^+(x + \lambda(y - x), y - x; 0)$

Proof:

Ad 1. $g^+(x, y; \mu)$ exists for every $\mu \in \mathbb{R}$ because $g(x, y; \bullet)$: $\mathbb{R} \to \mathbb{R}$ is a convex function. Differentiation therefore yields

$$c^+(x, y; \mu) = \frac{-g^+(x, y; \mu)}{g^2(x, y; \mu)} (x + \mu y) + \frac{y}{g(x, y; \mu)}$$

and with $\mu = 0$ we get

$$c^+(x, y; 0) = \frac{-g^+(x, y; 0)}{g^2(x, y; 0)} x + \frac{y}{g(x, y; 0)} = -g^+(x, y; 0) \frac{x}{\|x\|^2} + \frac{y}{\|x\|}.$$

Ad 2. 1. shows that $c^+(x, y; 0)$, x are linearly independent, because x, y are linearly independent. But $x = \|x\| c(x, y; 0)$, hence $c^+(x, y; 0)$, $c(x, y; 0)$ are linearly independent.

Ad 3. $x \perp y$ is defined by $\|x + \mu y\| \geq \|x\|$ for all $\mu \in \mathbb{R}$, but this implies that $\mu = 0$ is an argument where the absolute minimum $\|x\|$ of $g(x, y; \bullet)$ is attained and therefore we must have $-g^+(x, y; 0) \leq 0$.

Ad 4. One easily can check that

$$c(x,y;\mu) = c\left(x,\tilde{y};\frac{\mu}{\eta(1-\kappa\mu)}\right)$$

for all $\mu\in\mathbb{R}$ with $\kappa\mu < 1$. Differentiation yields

$$c^{+}(x,y;\mu) = c^{+}\left(x,\tilde{y};\frac{\mu}{\eta(1-\kappa\mu)}\right)\frac{1}{\eta(1-\kappa\mu)^2}$$

and for $\mu = 0$ we get the desired equation. ■

Ad 5. A trivial computation.

Lemma 2. $x\perp c^{+}(x,y;0)$ *for every pair of linearly independent vectors* $x, y\in X$.

Proof:
We only have to show that $\|x+\lambda c^{+}(x,y;0)\| \geq \|x\|$ for every $\lambda\in\mathbb{R}$. For shorter notation we will write $c(\delta) := c(x,y;\delta)$. Let $\delta > 0$ and $\mu\notin[0, 1]$, we then get

$$\|(1-\mu)c(\delta) + \mu c(0)\| \geq (1-\mu)\|c(\delta)\| + \mu\|c(0)\| = 1 = \|c(0)\|$$

or equivalently

$$\left\|c(\delta) + (-\mu)\delta\left(\frac{c(\delta)-c(0)}{\delta}\right)\right\| \geq \|c(0)\|.$$

The last inequality says that for all $\lambda\notin[-\delta, 0]$

$$\left\|c(\delta) + \lambda\left(\frac{c(\delta)-c(0)}{\delta}\right)\right\| \geq \|c(0)\|.$$

If we choose an $\varepsilon > 0$, then for every $0 < \delta < \varepsilon$ and every $\lambda\notin[-\varepsilon, 0]$ we get

$$\left\|c(\delta) + \lambda\left(\frac{c(\delta)-c(0)}{\delta}\right)\right\| \geq \|c(0)\|.$$

Taking the limit $\delta\to 0$ yields

$$\|c(0) + \lambda c^{+}(0)\| \geq \|c(0)\|$$

for every $\lambda\notin[-\varepsilon, 0]$. But ε was arbitrary and therefore

$$\|x + \lambda c^{+}(0)\| \geq \|x\|$$

for all $\lambda\in\mathbb{R}$. ■

Lemma 3. *Let* $\| \|_1$, $\| \|_2$ *be two norms on* X *which determine the same orthogonality relation* $\perp$ on X (*i.e.* $\perp_1 = \perp_2 =: \perp$). *Let* $g_i(x, y; \bullet)$, $c_i(x, y; \bullet)$ *be the functions with respect to* $\| \|_i$, $i = 1, 2$, *then*

$$c_2^+(x,y;0) = \frac{\|x\|_1}{\|x\|_2} c_1^+(x,y;0).$$

Proof:
By Lemma 1 we have

$$c_1^+(x,y;0) = -g_1^+(x,y;0)\frac{x}{\|x\|_1^2} + \frac{y}{\|x\|_1} \quad \text{and}$$

$$c_2^+(x,y;0) = -g_2^+(x,y;0)\frac{x}{\|x\|_2^2} + \frac{y}{\|x\|_2}$$

Substituting

$$\tilde{y} := c_1^+(x,y;0) = \frac{1}{\|x\|_1}\left(y + \frac{-g_1^+(x,y,;0)}{\|x\|_1}x\right)$$

in these two equations yields (according to Lemma 1.4)

$$c_1^+(x,\tilde{y};0) = \frac{\tilde{y}}{\|x\|_1} \quad \text{and} \quad c_2^+(x,\tilde{y};0) = \frac{1}{\|x\|_1}c_2^+(x,y;0)$$

From this we get (by Lemma 1.1)

$$\frac{1}{\|x\|_1}c_2^+(x,y;0) = c_2^+(x,\tilde{y};0) = -g_2^+(x,\tilde{y};0)\frac{x}{\|x\|_2^2} + \frac{\tilde{y}}{\|x\|_2}$$

$$= -g_2^+(x,\tilde{y};0)\frac{x}{\|x\|_2^2} + \frac{1}{\|x\|_2}c_1^+(x,y;0).$$

By Lemma 2 we have $x \perp \tilde{y}$ and Lemma 1.3 yields therefore

$$-g_2^+(x,\tilde{y};0) \le 0.$$

If we substitute $\tilde{y} := c_2^+(x,y;0)$ in our starting equations, we get the analogous equation

$$\frac{1}{\|x\|_2}c_1^+(x,y;0) = -g_1^+(x,\tilde{y};0)\frac{x}{\|x\|_1^2} + \frac{1}{\|x\|_1}c_2^+(x,y;0),$$

with the analogous statement that

$$-g_1^+(x,\tilde{y};0) \le 0.$$

Adding these last two equations yields

$$-g_2^+(x,\tilde{y};0)\frac{x}{\|x\|_2^2}+-g_1^+(x,\tilde{y};0)\frac{x}{\|x\|_1^2}=0,$$

which implies that both coefficients are zero, because they are both ≤ 0. Now we are ready, because of this we have

$$\frac{1}{\|x\|_1}c_2^+(x,y;0)=\frac{1}{\|x\|_2}c_1^+(x,y;0),$$

what we wanted. ■

Proof of the theorem:
Let $\| \|_1, \| \|_2$ be two norms on X, which define the same orthogonality relation $\perp$ on X and $x, y \in X$ linearly independent, then we will show that

$$\frac{\|y\|_1}{\|y\|_2}=\frac{\|x\|_1}{\|x\|_2}.$$

Our two curves are related in the following form

$$c_2(x,y-x;\lambda)=\frac{g_1(x,y-x;\lambda)}{g_2(x,y-x;\lambda)}c_1(x,y-x;\lambda)$$

and therefore we get by differentiation

$$c_2^+(x,y-x;\lambda)=\left(\frac{g_1}{g_2}\right)^+(x,y-x;\lambda)\,c_1(x,y-x;\lambda)$$
$$+\left(\frac{g_1}{g_2}\right)(x,y-x;\lambda)\,c_1^+(x,y-x;\lambda).$$

Lemma 1.5, Lemma 3 and again Lemma 1.5 yield

$$c_2^+(x,y-x;\lambda)=c_2^+(x+\lambda(y-x),y-x;0)$$
$$=\frac{\|x+\lambda(y-x)\|_1}{\|x+\lambda(y-x)\|_2}c_1^+(x+\lambda(y-x),y-x;0)$$
$$=\frac{\|x+\lambda(y-x)\|_1}{\|x+\lambda(y-x)\|_2}c_1^+(x,y-x;\lambda)$$

i.e. $c_2^+(x,y-x;\lambda)$, $c_1^+(x,y-x;\lambda)$ are linearly dependent. On the other side we have according to Lemma 1.2, that $c_1(x,y-x;\lambda)$, $c_1^+(x,y-x;\lambda)$

are linearly independent and hence we conclude, that

$$\left(\frac{g_1}{g_2}\right)^+ (x, y - x; \lambda) = 0$$

for all $\lambda \in \mathbb{R}$. From this we get (by the théorème des accroissements finis [4])

$$\left| \frac{g_1(x, y - x; 1)}{g_2(x, y - x; 1)} - \frac{g_1(x, y - x; 0)}{g_2(x, y - x; 0)} \right| \leq 0$$

or, what is the same

$$\frac{\| y \|_1}{\| y \|_2} = \frac{\| x \|_1}{\| x \|_2}.$$

This implies the existence of a number $\sigma \in \mathbb{R}$, such that for all $x \in X$

$$\| x \|_1 = \sigma \| x \|_2.$$

The implication from proportionality of the two norms on equality of there associated orthogonality relations is trivial. ■

References

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