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On Ring-like Structures Occurring in Axiomatic Quantum Mechanics*

By

D. Dorninger, H. Länger, and **M. Maczyński**

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Abstract

Ring-like partial algebras are studied that correspond to bounded lattices with an involutory antiautomorphism and give rise to certain kinds of quantum logics. Various extensions of the partial algebras to total algebras are investigated. Implications of associativity and distributivity are discussed and relations to the structures of the corresponding lattices are derived.

1. Introduction

A study of ring-like structures which are generalizations of Boolean rings has been initiated in [2] and later developed in [1] and [5]. The motivation for this study is to find a most general framework for developing axiomatic quantum mechanics. It is well-known that usually orthomodular lattices and generalizations of such lattices are used as models for quantum logics. In the lattice-theoretical approach, however, only the lattice meet and the complement have direct logical (and hence physical)

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interpretations; the lattice join cannot be interpreted directly (unlike the join in Boolean algebras). Hence it was suggested in [2] to consider a ring-theoretical approach to quantum logic, where the ring operations correspond to meet (ring multiplication) and to symmetric difference Δ (ring addition in Boolean rings), respectively. The symmetric difference gives rise to a pseudometric by defining

$$d(a,b) := p(a \, \Delta \, b)$$

for any subadditive measure p, which allows for a physical interpretation. In the present study we will develop this idea further by starting with partial generalized Boolean quasirings (pGBQRs) where only a partial addition $\oplus$ with 0 and 1 is assumed (the latter corresponding to lattice complement) and where the only total operation is multiplication (corresponding to lattice meet). We show how this partial algebraic operation $\oplus$ can be extended to a total operation of addition $+$ and we study various possibilities of this extension (corresponding to various generalizations of the operation of symmetric difference). We give sufficient and necessary conditions for these operations to be associative and relate the associativity of our structures to the distributivity of the induced lattices. In particular, a simple characterization is derived for the case that the induced lattice is a de Morgan algebra. Moreover, we introduce an operation which gives rise to the notion of an implication in pGBQRs, and we study the various kinds of distributivity due to the different operations that occur.

In [2] a generalized Boolean quasiring (GBQR) was introduced as an algebra $(R, +, \cdot)$ of type (2,2) which contains two elements 0 and 1 such that the following laws (1)–(8) hold:

$$x + y = y + x \tag{1}$$

$$x + 0 = x \tag{2}$$

$$(xy)z = x(yz) \tag{3}$$

$$xy = yx \tag{4}$$

$$xx = x \tag{5}$$

$$x0 = 0 \tag{6}$$

$$x1 = x \tag{7}$$

$$1 + (1 + xy)(1 + x) = x \tag{8}$$

It was shown in [2] that if one defines in R $x \vee y := 1 + (1 + x)(1 + y)$, $x \wedge y := x \cdot y$ and $x' := 1 + x$ the algebra $\mathbf{L}(R) := (R, \vee, \wedge, ')$ is a bounded lattice with an involutory antiautomorphism$'$. On the other hand, if one starts with a bounded lattice $(L, \vee, \wedge, ')$ with an involutory antiautomorphism$'$ and defines $x + y := (x \vee y) \wedge (x \wedge y)'$ and $xy := x \wedge y$, then $\mathbf{R}(L) := (L, +, \cdot)$ is a GBQR, though of a certain type: For all x, y the equation

$$(1 + (1 + x)(1 + y))(1 + xy) = x + y \tag{9}$$

is fulfilled. In [1] a GBQR R in which equation (9) holds for all $x, y, \in R$ was called *uniquely representable*, because, as proved in [2] for GBQRs with (9), $\mathbf{R}(\mathbf{L}(R)) = R$ and $\mathbf{L}(\mathbf{R}(L)) = L$.

However, as we will see this result is only a special case of a property of partial GBQRs due to a certain extension of the partial operation $\oplus$.

As far as uniquely representable GBQRs are concerned, in the following we will make use of the two properties below (cf [1], [2]): For all $x, y \in R$, R a uniquely representable GBQR,

$$x + y = (1 + x) + (1 + y) \tag{10}$$

$$x(x + y) = x(1 + xy) = x + xy \tag{11}$$

2. Partial GBQRs and Extensions

Definition 2.1. A partial algebra $(R, \oplus, \cdot)$ of type (2,2) is called a *partial generalized Boolean quasiring* (shortly, pGBQR) if there exist elements 0 and 1 of R such that $\oplus : \{0, 1\} \times R \to R$ and $\cdot : R \times R \to R$ and the following hold:

$$0 \oplus x = x \tag{2'}$$

$$(xy)z = x(yz) \tag{3'}$$

$$xy = yx \tag{4'}$$

$$xx = x \tag{5'}$$

$$x0 = 0 \tag{6'}$$

$$x1 = x \tag{7'}$$

$$1 \oplus (1 \oplus xy)(1 \oplus x) = x \tag{8'}$$

If we define for a pGBQR $(R, \oplus, \cdot)$

$$\begin{aligned} x \vee y &:= 1 \oplus (1 \oplus x)(1 \oplus y) \\ x \wedge y &:= xy \\ x' &:= 1 \oplus x \end{aligned}$$

for all $x, y \in R$ and put $\mathbf{L}(R) := (R, \vee, \wedge, ', 0, 1)$, then one can show in the very same way as for GBQRs (cf. [2]):

Theorem 2.1. $\mathbf{L}(R)$ *is a bounded lattice with an involutory antiautomorphism.*

On the other hand, let $L = (L, \vee, \wedge, ', 0, 1)$ be a bounded lattice with an involutory antiautomorphism. If we now define

$$1 \oplus x := x'$$
$$0 \oplus x := x$$
$$xy := x \wedge y$$

and put $\mathbf{R}(L) := (L, \oplus, \cdot)$, then one can prove by using the same arguments as for GBQRs (cf. [2]):

Theorem 2.2. $\mathbf{R}(L)$ *is a pGBQR.*

Moreover, as for GBQRs we have

Theorem 2.3. $\mathbf{L}(\mathbf{R}(L)) = L$ *and* $\mathbf{R}(\mathbf{L}(R)) = R$.

The proof of Theorem 2.3 follows the same lines as the respective proof for GBQRs in [2].

Definition 2.2. Let $(R, +, \cdot)$ be a GBQR. The *core* of R is the partial algebra $(R, \oplus, \cdot)$ of type (2,2) defined by

$$0 \oplus x := x$$
$$1 \oplus x := 1 + x$$

Obviously, the core of every GBQR is a pGBQR. Conversely, if $(R, \oplus, \cdot)$ is a pGBQR, then we can extend $(R, \oplus, \cdot)$ to a GBQR $(R, +, \cdot)$ by defining

$$0 + x = x + 0 := 0 \oplus x,$$
$$1 + x = x + 1 := 1 \oplus x, \text{ and}$$
$$\text{for } x, y \in R \backslash \{0, 1\} \quad x + y = y + x \in R.$$

Clearly, the core of $(R, +, \cdot)$ is exactly $(R, \oplus, \cdot)$.

In the following any extension $+$ of $\oplus$ is always meant in the way described above and x' will be short for $1 + x = 1 \oplus x$.

The canonical examples for extensions of $\oplus$ to a full operation $+$ are

$$x +_1 y := 1 \oplus (1 \oplus x(1 \oplus y))(1 \oplus (1 \oplus x)y) \text{ and}$$
$$x +_2 y := (1 \oplus (1 \oplus x)(1 \oplus y))(1 \oplus xy).$$

If $\mathbf{L}(R)$ is a Boolean algebra, both $+_1$ and $+_2$ correspondend to the symmetric difference $\triangle$. The operation $+_1$ is suggested by the identity

$x\triangle y = (x \wedge y') \vee (x' \wedge y) = ((x \wedge y')' \wedge (x' \wedge y)')'$ and $+_2$ arises by $x\triangle y = (x \vee y) \wedge (x' \vee y') = (x' \wedge y')' \wedge (x \wedge y)'$. The extension $+_2$ yields a uniquely representable GBQR (as defined in Section 1).

Next we discuss how $+_1$, $+_2$ and $+$ in general are related to each other. For this end we take into account that two important features of $\triangle$ are $x'\triangle y' = x\triangle y$ and $x' \wedge y \leqslant x\triangle y \leqslant x \vee y = (x' \wedge y')'$ which give rise to the assumptions

(i) $x' + y' = x + y$ and
(ii) $x'y \leq x + y \leq (x'y')'$

For pGBQRs and GBQRs $x \leq y$ of course means $xy = x$. As one can see easily (by calculations within the respective GBQRs or by means of the underlying lattices $\mathbf{L}(R)$) both assumptions are met by $+_1$ and $+_2$.

In the following we will frequently make use of the property that the algebra $\mathbf{L}(R)$ associated with a GBQR is a lattice. The operations $\vee$ and $\wedge$ within results and proofs will always refer to this lattice. Moreover we will take advantage of the relation $x \perp y$ (x is *orthogonal* to y) as defined for lattices (namely $x \perp y \Leftrightarrow x \leq y'$). As for the commuting-relation xCy this needs some care. Let R be an arbitrary GBQR and $x, y \in R$. According to [1] we define that x *commutes* with y by

$$xCy :\Leftrightarrow y(1 + x) = y + yx$$

Theorem 2.4. *Let* R *be a GBQR that fulfills the assumptions (i) and (ii). Then* $x +_1 y \leq x + y \leq x +_2 y$ *for all* $x, y \in$ R. *Moreover, if there exist two elements* $a, b \in$ R *with* $a \neq b$ *such that* $a = a'$ *and* $b = b'$ *or if there exist elements* $c, d \in$ R *with* $cd' = 0$ *that do not commute then* $+_1$ *and* $+_2$ *are not equal.*

Proof: Since $'$ is an involutory antiautomorphism $x + y \geq x'y$ implies $1 + (x + y) \leq 1 + x'y$ and $x + y = x' + y' \geq (x')'y' = xy'$ implies $1 + (x + y) \leq 1 + xy'$. Therefore $1 + (x + y) \leq (1 + x'y)(1 + xy')$, hence $x + y \geq x +_1 y$.
On the other hand $x + y \leq 1 + x'y'$ and $x + y = x' + y' \leq 1 + (x')'(y')'$ yields $x + y \leq (1 + x'y')(1 + xy) = x +_2 y$.
Now we assume that there exist $a, b \in R$ with $a \neq b$ such that $a = a'$, $b = b'$ and $+_1 = + = +_2$. Then $a +_1 b = 1 + (1 + ab')(1 + a'b) = 1 + (1 + ab) = ab$ and $a +_2 b = (1 + a'b')(1 + ab) = 1 + ab$. Therefore $ab = 1 + ab = 1 + a'b'$, i.e. $a \wedge b = a \vee b$, hence $a = b$.

If there are $c, d \in R$ which do not commute such that $cd' = 0$ and $+_1 = + = +_2$, then $c +_1 d = c +_2 d$ implies $1 + (1 + cd')(1 + c'd) =$

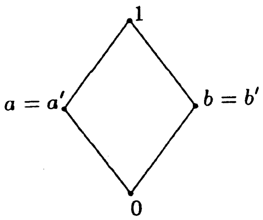

Fig. 2.1

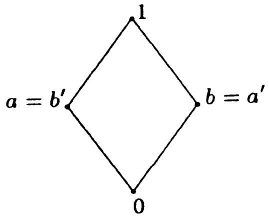

Fig. 2.2

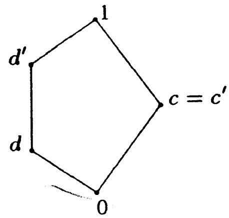

Fig. 2.3

$c'd = (1 + c'd')(1 + cd)$, wherefrom $d(1 + c) = dc'd = d(1 + c'd')(1 + cd) = d(1 + cd) = d + cd$ follows by (8) and (11). Therefore we would have cCd, a contradiction. □

Figures 2.1 and 2.3 show examples of lattices $\mathbf{L}(R)$ of GBQRs R for which $+_1 \neq +_2$. For the GBQR R whose lattice is illustrated in Fig. 2.2 the operations $+_1$ and $+_2$ coincide.

Theorem 2.5. *For every pair of elements x, y of an arbitrary GBQR R $x +_1 y \leq x +_2 y$. If $x \perp y$ or if $x \leq y$ equality holds. In particular, if $\mathbf{L}(R)$ is a chain, then $x +_1 y = x +_2 y$ for all $x, y \in R$.*

Proof: (8) implies $x \leq 1 + x'y$, $y \leq 1 + xy'$, $x' \leq 1 + xy'$, and $y' \leq 1 + x'y$. Therefore $1 + xy \geq 1 + (1 + x'y)(1 + xy') = x +_1 y$ and $1 + x'y' \geq 1 + (1 + xy')(1 + x'y) = x +_1 y$, wherefrom we can conclude $x +_2 y = (1 + x'y')(1 + xy) \geq x +_1 y$. If $x \perp y$ then $x \leq y'$ and equivalently $x' \geq y$. Thus $x +_1 y = 1 + (1 + x)(1 + y) \geq (1 + x'y')(1 + xy) = x +_2 y$. If $x \leq y$ then $x +_2 y = (1 + y')(1 + x) = yx'$ because of $x' \geq y'$, and $1 + (x +_1 y) = (1 + xy')(1 + (x +_2 y))$. Therefore $1 + (x +_1 y) \leq 1 + (x +_2 y)$, hence $x +_1 y \geq x +_2 y$. □

3. Associativity of Addition

We now reduce our investigations to the case that $+$ is either $+_1$ or $+_2$. For other kinds of extensions of $\oplus$ similar results can be obtained (if $+$ fulfils appropriate assumptions). As for the associativity of $+_1$ and $+_2$ it even suffices only to study $+_2$, because

Lemma 3.1. *For a pGBQR the extension $+_1$ is associative iff $+_2$ is associative. In this case $+_1$ and $+_2$ are equal.*

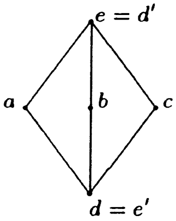

Fig. 3.1

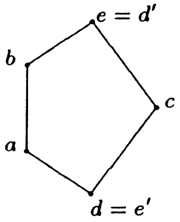

Fig. 3.2

Proof: For $i, j \in \{1,2\}$ and $i \neq j$ $a' +_i b' = a +_i b$ and $a +_i b = (a' +_j b)'$. Therefore, if we assume $+_i$ to be associative, we obtain for $x, y, z \in R$: $x +_j (y +_j z) = (x' +_i (y' +_i z)')' = (x +_i ((1 +_i y) +_i z))' = (1 +_i (x +_i (y +_i z)))' = x +_i (y +_i z)$. From this we can conclude $x +_j (y +_j z) = (x +_j y) +_j z$. Moreover, putting $z = 0$ in $x +_j (y +_j z) = x +_i (y +_i z)$ shows $x +_j y = x +_i y$. □

Lemma 3.2. *Let* $(R, +_2, \cdot)$ *be a GBQR such that* $+_2$ *is associative. Then* $\mathbf{L}(R)$ *must not contain one of the following sublattices* $\{a, b, c, d, e\}$ *illustrated in Fig. 3.1 and Fig. 3.2 for which we assume* $d' = e$.

Proof: Let $+$ be $+_2$. If $\mathbf{L}(R)$ contains a sublattice isomorphic to that in Fig. 3.1 then

$$(a + b) + c = (e \vee c) \wedge (e' \vee c') = e \wedge c' = c' \neq a' = e \wedge a' = \\ = (a \vee e) \wedge (a' \vee e') = a + (b + c).$$

If $\mathbf{L}(R)$ contains a sublattice isomorphic to that in Fig. 3.2 then

$$(a + c) + b = (e \vee b) \wedge (e' \vee b') = e \wedge b' = b' \neq a' = e \wedge a' = \\ = (a \vee e) \wedge (a' \vee e') = a + (c + b).$$

□

Conjecture 3.1. Associativity of $+_2$ in a GBQR $(R, +_2, \cdot)$ implies distributivity of $\mathbf{L}(R)$.

To the end of this section we will assume $\mathbf{L}(R)$ to be distributive, i.e. $\mathbf{L}(R)$ is a de Morgan algebra. (In this case a normal form system for the polynomials over $\mathbf{L}(R)$ is known, which can be used to prove the equality of terms (cf. [3]).) Moreover, $+$ will always be assumed to be $+_2$.

Lemma 3.3. *Let* R *be a uniquely representable GBQR such that* $\mathbf{L}(R)$ *is distributive. Then* $+$ *is associative iff* $(1 + y) + z = 1 + (y + z)$ *for all*

$y, z \in R$, *which, in terms of lattice operations means*

$$(y' \vee z) \wedge (y \vee z') = (y' \wedge z') \vee (y \wedge z). \tag{12}$$

Proof: If equation (12) holds

$$\begin{aligned}(x+y)+z &= [((x \vee y) \wedge (x' \vee y')) \vee z] \wedge [(x \vee y) \wedge (x \wedge y)' \wedge z]' = \\ &= (x \vee y \vee z) \wedge (x' \vee y' \vee z) \wedge [((x' \wedge y') \vee (x \wedge y)) \vee z'] = \\ &= (x \vee y \vee z) \wedge (x' \vee y' \vee z) \wedge (x' \vee y \vee z') \wedge (x \vee y' \vee z').\end{aligned}$$

$x + (y + z)$ is obtained by interchanging x and z within the last expression which exactly yields the same terms. Hence $(x+y)+z = x+(y+z)$. □

Lemma 3.4. *Let R be a uniquely representable GBQR such that* $\mathbf{L}(R)$ *is distributive. Then* $(1+y)+z = 1+(y+z)$ *for all* $y, z \in R$ *iff* $yy' \perp zz'$ *for every y and z.*

Proof: $(1+y)+z = 1+(y+z)$ is equivalent to (12). From (12) it follows

$$y \wedge y' \leq (y' \vee z) \wedge (y \vee z') = (y' \wedge z') \vee (y \wedge z) \leq z \vee z'.$$

Conversely, if $y \wedge y' \perp z \wedge z'$ for all y, z the inequalities $y \wedge y' \leq z \vee z'$ and $z \wedge z' \leq y \vee y'$ imply

$$\begin{aligned}(y' \vee z) \wedge (y \vee z') &= (y' \wedge y) \vee (y' \wedge z') \vee (z \wedge y) \vee (z \wedge z') \leq \\ &\leq (y' \vee y) \wedge (y' \vee z) \wedge (z' \vee y) \wedge (z' \vee z) \\ &= (y' \wedge z') \vee (y \wedge z)\end{aligned}$$

whence (12) follows. □

Definition 3.1. Let L be a lattice with an involutory antiautomorphism $'$ and $OK(L) := \{x \wedge x' | x \in L\}$. Then - as can be seen easily - $OK(L)$ is an order ideal which can also be characterized by $OK(L) = \{x \in L | x \perp x\}$. Therefore $OK(L)$ will be called the *orthogonal kernel* of L.

Moreover, if for a subset A of L $a \perp b$ for all distinct $a, b \in A$, A will be called orthogonal.

Theorem 3.1. *Let R be a uniquely representable GBQR such that* $\mathbf{L}(R)$ *is distributive. Then the following are equivalent:*

(i) $+_1$ *is associative.*
(ii) $+_2$ *is associative.*

(iii) $+_1$ *equals* $+_2$.
(iv) $OK(\mathbf{L}(R))$ *is orthogonal.*

Proof: The equivalence of (i), (ii) and (iii) is due to Lemmata 3.1 and 3.3 (dual form of (12)) and that (ii) and (iv) are equivalent follows by Lemmata 3.3 and 3.4. □

4. Distributivity in GBQRs

Definition 4.1. An (arbitrary) GBQR $(R, +, \cdot)$ is called *distributive*, if $\cdot$ is distributive with respect to $+$.

Lemma 4.1. *Let* $+$ *be an arbitraty extension of the operation* $\oplus$ *of a pGBQR* $(R, \oplus, \cdot)$. *If* $y(1+x) = y + xy$ *for all* $x, y \in R$ *then* $\mathbf{L}(R)$ *is a Boolean algebra.*

Proof: Assume $y(1+x) = y + xy$ for any $x, y \in R$, which means that xCy for all $x, y \in R$. Then $(1+xy) + (x+xy) = (1+xy) + x(1+xy) = (1+xy)(1+x) = 1+x$ by (8). Substituting $1+xy$ for x in

$$(1+xy) + (x+xy) = 1+x \tag{13}$$

and again applying (13) we obtain that $(1+(y+xy)) + ((1+xy) + (y+xy)) = (1+(y+xy)) + (1+y)$ equals $1+(1+xy) = xy$. Thus, if we put $x = 0$ we infer $(1+y)+(1+y) = 0$ for every $y \in R$. From this we can conclude that R is of characteristic 2, i.e. $x + x = 0$ for all $x \in R$. $x + x = 0$ implies $x(1+x) = xx' = x \wedge x' = 0$ and $x \vee x' = 1 + xx' = 1$. Therefore $\mathbf{L}(R)$ is an ortholattice. Moreover, $x \leq y$ yields $x \vee (y \wedge x') = 1 + (1+x)(1+yx') = 1 + (x' + yx') = 1 + x'(1+y) = 1 + y' = y$, which shows that $\mathbf{L}(R)$ is orthomodular. Since xCy implies $y \wedge x' = y \wedge (y' \vee x')$, $\mathbf{L}(R)$ is a Boolean algebra (cf. [4]). □

Theorem 4.1. *Let* R *be an arbitrary GBQR* $(R, +, \cdot)$ *such that* $+$ *fulfils the assumptions (i)* $x' + y' = x + y$ *and (ii)* $x'y \leq x + y \leq 1 + x'y'$. *Then* R *is distributive iff* $\mathbf{L}(R)$ *is a Boolean algebra, in which case* R *is a Boolean ring and* $+ = +_1 = +_2$.

Proof: If $\cdot$ is distributive with respect to $+$, $y(1+x) = y + xy$ for all $x, y \in R$, hence $\mathbf{L}(R)$ is a Boolean algebra by Lemma 4.1.
Conversely, assume $\mathbf{L}(R)$ is a Boolean algebra. Then according to Theorem 2.4 (i) and (ii) imply $x +_1 y \leq x +_2 y$. Since $\mathbf{L}(R)$ is a Boolean algebra, we obtain $x +_1 y = x + y = x +_2 y$. Because $x + x = 0$ for all $x \in R$, R is a Boolean quasiring (in the sense of [2]), and as shown in [2],

Boolean quasirings for which $\mathbf{L}(R)$ is a Boolean algebra are Boolean rings. $\square$

Let $(R, \oplus, \cdot)$ be a pGBQR. We define on R a new binary operation, denoted by $*$, as follows:

$$x * y := 1 \oplus x(1 \oplus y)$$

for all $x, y \in R$. This operation in $\mathbf{L}(R)$ is equivalent to $x * y = (x \wedge y')' = x' \vee y$ and is also denoted by $x \to y$ (the operation of *implication*).

Definition 4.2. We say that $(R, \oplus, \cdot)$ is $*$-*distributive* if $*$ is left-distributive with respect to $\cdot$, i.e.

$$x * (yz) = (x * y)(x * z)$$

for all $x, y, z \in R$.

We have the following theorem:

Theorem 4.2. *A GBQR R is $*$-distributive iff $\mathbf{L}(R)$ is distributive.*

Proof: For all $x, y, z \in R$ we have

$$x * (yz) = x' \vee (y \wedge z)$$
$$(x * y)(x * z) = (x' \vee y) \wedge (x' \vee z)$$

$\square$

The operation $*$ is in some sense dual to $\cdot$ since for all $x, y \in R$ we have

$$x * y = 1 \oplus x(1 \oplus y) = (xy')'$$
$$xy = 1 \oplus (x * (1 \oplus y)) = (x * y')'$$

Interchanging $*$ with $\cdot$ does not violate the identity.

Let us point out that the operation $*$ seems more natural to be used in the theory of GBQRs than the operation $\vee$ (lattice join) because $x * y$ can be interpreted as implication $x \to y$ which has some meaning when we pass to quantum logic whereas the operation $\vee$ has no direct interpretation in the framework of quantum logic. In quantum logic, $*$-distributivity can be naturally interpreted as

$$p \to (q \wedge r) \Leftrightarrow (p \to q) \wedge (p \to r)$$

whereas lattice distributivity has no direct quantum logic interpretation (though both properties are equivalent in GBQRs).

References

[1] Dorninger, D.: *Sublogics of ring-like quantum logics*. Tatra Mt. Math. Publ. (to appear).

[2] Dorninger, D., Länger, H., Maczyński, M.: *The logic induced by a system of homomorphisms and its various algebraic characterizations*. Demonstr. Math. **30** (1997), 215–232.

[3] Dorninger, D., Schweigert, D.: *Zur Darstellung von Polynomen auf De Morgan Algebren*. Czechosl. Math. J. **30 (105)** (1980), 65–70.

[4] Kalmbach, G.: *Orthomodular lattices*. Academic Press, London – New York, 1983.

[5] Länger, H.: *Generalizations of the correspondence between Boolean algebras and Boolean rings to orthomodular lattices*. Tatra Mt. Math. Publ. (to appear).

Authors' addresses: Dietmar Dorninger and Helmut Länger, Technische Universität Wien, Institut für Algebra und Diskrete Mathematik, Wiedner Hauptstraße 8–10, A–1040 Wien; Maciej Maczyński, Politechnika Warszawska, Instytut Matematyki, Plac Politechniki 1, PL 00–661 Warszawa.