Elliptic Means and Their Generalizations

By

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Abstract

In this note we introduce a one-parameter family of homogeneous means strictly related to ellipses. Each member of the family is a weighted power mean, and only one of them is both symmetric and quasi-arithmetic. Geometric interpretations are given, and higher-dimensional counterparts of these means are defined. Iterations of some mean-type mappings and some functional equations are considered.

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1. Introduction

Let $I \subset \mathbb{R}$ be an interval and $k \in \mathbb{N}$, $k \geq 2$, fixed. A function $M: I^k \to \mathbb{R}$ is said to be a *mean* if

$$\min(x_1, \ldots, x_k) \leq M(x_1, \ldots, x_k) \leq \max(x_1, \ldots, x_k), \quad x_1, \ldots, x_k \in I.$$

A mean M is called *strict* if $\min(x_1, \ldots, x_k) < \max(x_1, \ldots, x_k)$ implies that the above inequalities are strict, and M is called *symmetric* if, for every permutation $\sigma: \{1, \ldots, k\} \to \{1, \ldots, k\}$,

$$M(x_1, \ldots, x_k) = M(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), \quad x_1, \ldots, x_k \in I.$$

For any continuous and strictly monotonic function $\varphi: I \to \mathbb{R}$ and a sequence $w = (w_1, \ldots, w_k)$, $w_1 > 0, \ldots, w_k > 0$, $w_1 + \cdots$

$+ w_k = 1$, the function $M_{k,w}^{\varphi} \colon I^k \to \mathbb{R}$,

$$M_{k,w}^{\varphi}(x_1,\ldots,x_k) := \varphi^{-1}(w_1\varphi(x_1) + \cdots + w_k\varphi(x_k)), \quad x_1,\ldots,x_k \in I,$$

is a strict mean, and it is called a *weighted quasi-arithmetic mean*. The function φ is referred to as a generator of the mean $M_{k,w}^{\varphi}$ and the numbers $w_1, \ldots, w_k$ as its weights. $M_{k,w}^{\varphi}$ is symmetric iff $M_{k,w}^{\varphi} = M_k^{\varphi}$ where

$$M_k^{\varphi}(x_1,\ldots,x_k) := \varphi^{-1}\left(\frac{\varphi(x_1) + \cdots + \varphi(x_k)}{k}\right), \quad x_1,\ldots,x_k \in I,$$

and M_k^{φ} is called a *quasi-arithmetic mean*.

A mean $M \colon (0,\infty)^k \to (0,\infty)$ is called *homogeneous* if

$$M(tx_1,\ldots,tx_k) = tM(x_1,\ldots,x_k), \quad t,x_1,\ldots,x_k > 0.$$

It is well known (cf. B. JESSEN [4], also G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA [2], p. 68) that a weighted quasi-arithmetic mean is homogeneous iff it is a weighted power mean, that is, there is an $r \in \mathbb{R}$ such that $M_{k,w}^{[\varphi]} = M_{k,w}^{[r]}$ where

$$M_{k,w}^{[r]}(x_1,\ldots,x_k) := \begin{cases} (w_1 x_1^r + \cdots + w_k x_k^r)^{1/r} & \text{for} \quad r \neq 0, \\ x_1^{w_1} \cdots x_k^{w_k} & \text{for} \quad r = 0. \end{cases}$$

Note that $M_k^{[0]} = G_k$ where G_k denotes the geometric mean.

In this note we show that the means $M_{2,w}^{[-2]}$ are strictly related to an ellipse, $M_{3,w}^{[-2]}$ to an ellipsoid, and $M_{k,w}^{[-2]}$ to a k-dimensional ellipsoid. In the first four sections we distinguish them by suitable symbols and formulate some of their properties as Propositions. In Section 5 we apply these Propositions to find all continuous solutions of a functional equation involving these means, closely related to the iteration of mean-type mappings.

2. Elliptic Means

We begin this section with the following quickly verifiable

Remark 1. *Let $p > 0$ be fixed. Then the function $E_p \colon (0,\infty)^2 \to (0,\infty)$,*

$$E_p(a,b) := ab\sqrt{\frac{p^2 + 1}{p^2 a^2 + b^2}}$$

is a mean.

We call the means E_p *elliptic* which is justified by the following

Geometric Interpretation. Consider an ellipse given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a, b > 0$, and take an arbitrary $p > 0$. It may be shown that the length $|OP|$, where O is the center of the ellipse and P is the intersection point of the ellipse and the half-line

$$y = px, \quad x \geq 0,$$

is given by $|OP| = E_p(a, b)$.

Proposition 1.

1. *For every $p > 0$, E_p is a weighted power mean $M_{2,w}^{[-2]}$ with weights*

$$w = \left(\frac{1}{p^2 + 1}, \frac{p^2}{p^2 + 1} \right);$$

   in particular, it is homogeneous, and the function $\varphi(t) = t^{-2} \ (t > 0)$ is a generator of this mean.
2. *E_p is symmetric iff $p = 1$;*
3. *E_p is quasi-arithmetic iff $p = 1$; moreover,*

$$E_1 = M_2^{[-2]}.$$

4. *For every $p > 0$, E_p is G_2-conjugate to the weighted square-root mean $M_{2,w^*}^{[2]}$ with weights*

$$w^* = \left(\frac{p^2}{p^2 + 1}, \frac{1}{p^2 + 1} \right),$$

   that is

$$G_2 \circ (E_p, M_{2,w^*}^{[2]}) = G_2,$$

   where G_2 denotes the geometric mean. G_2 is the unique continuous mean which is invariant with respect to the mean-type mapping $(E_p, M_{2,w^*}^{[2]}): (0, \infty)^2 \to (0, \infty)^2$; moreover, the sequence of iterates of the mean-type mapping $(E_p, M_{2,w^*}^{[2]})$ converges to the mean-type mapping (G_2, G_2) in $(0, \infty)^2$.
5. *$\lim_{p \to 0} E_p(x, y) = x$ and $\lim_{p \to \infty} E_p(x, y) = y$ for all $x, y > 0$.*

Proof. Parts 1–3 and 5 are not hard to verify. Part 4 is a consequence of some more general facts (the conjugate and invariant means were considered in [5] and [6], cf. also [1]).   □

Remark 2. *It can be readily shown that the following commutation relation* (*involving a parameter transformation*) *holds*:

$$E_p(a, b) = E_{1/p}(b, a), \quad p, a, b > 0.$$

3. Ellipsoidal Means

Let $p, q > 0$ be fixed. Then the function $E_{p,q} \colon (0, \infty)^3 \to (0, \infty)$,

$$E_{p,q}(a, b, c) := abc \sqrt{\frac{p^2 + q^2 + 1}{b^2 c^2 + p^2 c^2 a^2 + q^2 a^2 b^2}}$$

is a mean. $E_{p,q}$ can be called an ellipsoidal mean because of the following

Geometric Interpretation. Consider an ellipsoid given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where $a, b, c > 0$, and take the half-line determined by the equations

$$y = px, \quad z = qx, \quad x \geq 0,$$

for some arbitrary $p, q > 0$. Calculations show that $E_{p,q}(a, b, c)$ is the length $|OP|$ where O is the center of the ellipsoid and P is the point of intersection of the ellipsoid and the half-line.

Proposition 2.

1. *For all $p, q > 0$, $E_{p,q}$ is a weighted power mean $M_{3,w}^{[-2]}$ with weights*

$$w = \left(\frac{1}{p^2 + q^2 + 1}, \frac{p^2}{p^2 + q^2 + 1}, \frac{q^2}{p^2 + q^2 + 1} \right);$$

   in particular, it is homogeneous, and the function $\varphi(t) = t^{-2}$ ($t > 0$) is a generator of this mean.
2. *$E_{p,q}$ is symmetric iff $p = q = 1$;*
3. *$E_{p,q}$ is quasi-arithmetic iff $p = q = 1$; moreover,*

$$E_{1,1} = M_3^{[-2]}.$$

4. G_3 *is the unique continuous mean which is invariant with respect to the mean-type mapping* $(E_{p,q}, K_{p,q}, M^{[2]}_{3,w^*})\colon (0,\infty)^3 \to (0,\infty)^3$ *where the mean* $K_{p,q}\colon (0,\infty)^3 \to (0,\infty)$ *is defined by*

$$K_{p,q}(a,b,c) := \sqrt{\frac{p^2 b^2 c^2 + q^2 a^2 b^2 + c^2 a^2}{p^2 b^2 + q^2 a^2 + c^2}},$$

$$w^* = \left(\frac{q^2}{p^2+q^2+1}, \frac{p^2}{p^2+q^2+1}, \frac{1}{p^2+q^2+1}\right)$$

that is

$$G_3 \circ (E_{p,q}, K_{p,q}, M^{[2]}_{3,w^*}) = G_3;$$

the sequence of iterates of the mean-type mapping $\left(E_{p,q}, K_{p,q}, M^{[2]}_{3,w^*}\right)$ *converges to the mean-type mapping* (G_3, G_3, G_3) *in* $(0,\infty)^3$.
   Moreover, $K_{p,q}$ *is symmetric iff* $p = q = 1$.
5. $\lim_{q\to 0} E_{p,q}(x,y,z) = E_p(x,y)$ *and* $\lim_{q\to\infty} E_{p,q}(x,y,z) = z$ *for all* $x, y, z > 0$.

Remark 3. *The following commutation relations* (*involving some parameter transformations*) *can be readily verified*:

$$E_{p,q}(a,b,c) = E_{q,p}(a,c,b) = E_{1/p,q/p}(b,a,c) = E_{q/p,1/p}(b,c,a)$$
$$= E_{p/q,1/q}(c,b,a) = E_{1/q,p/q}(c,a,b)$$

for all $a, b, c, p, q > 0$.

4. The General k-Dimensional Case

Let $k \in \mathbb{N}$, $k \geq 2$, and $p_1, \ldots, p_{k-1} > 0$ be fixed. Then the function $E_{p_1,\ldots,p_{k-1}}\colon (0,\infty)^k \to (0,\infty)$ defined by

$$E_{p_1,\ldots,p_{k-1}}(a_1,\ldots,a_k)$$

$$:= a_1 \cdots a_k \sqrt{\frac{p_1^2 + \cdots + p_{k-1}^2 + 1}{p_1^2 a_2^2 \cdots a_k^2 + \cdots + p_{k-1}^2 a_1^2 \cdots a_{k-2}^2 a_k^2 + a_1^2 \cdots a_{k-1}^2}}$$

$$\times \left(E_{p_1,\ldots,p_{k-1}}(a_1,\ldots,a_k) := \left(\prod_{i=1}^{k} a_i\right) \sqrt{\frac{\sum_{i=1}^{k-1} p_i^2 + 1}{\sum_{j=1}^{k-1} p_j^2 \prod_{i\neq j}^{k} a_i^2 + \prod_{i=1}^{k-1} a_i^2}} \right)$$

is a mean. This mean may be referred to as k-dimensional ellipsoidal mean by an analogous geometric interpretation as in the previous cases.

Proposition 3.

1. *For all $p_1, \ldots, p_{k-1} > 0$, $E_{p_1,\ldots,p_{k-1}}$ is a weighted power mean $M_{k,w}^{[-2]}$ with weights*

$$w = \left(\frac{1}{\sum_{i=1}^{k-1} p_i^2 + 1}, \frac{p_1^2}{\sum_{i=1}^{k-1} p_i^2 + 1}, \ldots, \frac{p_{k-1}^2}{\sum_{i=1}^{k-1} p_i^2 + 1} \right);$$

   in particular, it is homogeneous, and the function $\varphi(t) = t^{-2}$ $(t>0)$ is a generator of this mean.
2. *$E_{p_1,\ldots,p_{k-1}}$ is symmetric iff $p_1 = \cdots = p_{k-1} = 1$;*
3. *$E_{p_1,\ldots,p_{k-1}}$ is quasi-arithmetic iff $p_1 = \cdots = p_{k-1} = 1$; moreover,*

$$E_{1,\ldots,1} = M_k^{[-2]}.$$

4. *G_k is the unique continuous mean which is invariant with respect to the mean-type mapping $(E_{1,\ldots,1}, K_1, \ldots, K_{k-1}) \colon (0,\infty)^k \to (0,\infty)^k$ where the means $K_l \colon (0,\infty)^k \to (0,\infty)$, $l = 1, \ldots, k-1$, are defined by*

$$K_l(a_1, \ldots, a_k) := \sqrt{\frac{(k-l) \sum_{j_1 < \cdots < j_l} \prod_{i \notin \{j_1,\ldots,j_l\}} a_i^2}{(l+1) \sum_{j_1 < \cdots < j_{l+1}} \prod_{i \notin \{j_1,\ldots,j_{l+1}\}} a_i^2}},$$

   that is

$$G_k \circ (E_{1,\ldots,1}, K_1, \ldots, K_{k-1}) = G_k;$$

   moreover, $K_{k-1} = M_k^{[2]}$, and the sequence of iterates of the mean-type mapping $(E_{1,\ldots,1}, K_1, \ldots, K_{k-2}, M_k^{[2]})$ converges to the mean-type mapping $(G_k, \ldots, G_k)$ in $(0,\infty)^k$.

5. An Application to a Functional Equation

HARUKI and RASSIAS [3] (cf. also [8]) posed the following

Problem 1. *Is it true that every continuous function $f \colon (0,\infty) \times (0,\infty) \to \mathbb{R}$ satisfying the functional equation*

$$f\left(\frac{2xy}{x+y}, \frac{x+y}{2} \right) = f(x,y), \quad x,y > 0,$$

is of the form

$$f(x,y) = F(xy), \quad x,y > 0,$$

where $F \colon (0,\infty) \to \mathbb{R}$ is a continuous function of a single variable?

The affirmative answer has been given by the second author (cf. [7]). Functional equations of the form

$$f(M(x,y), N(x,y)) = f(x,y), \quad x,y \in I,$$

where $M, N: I^2 \to I$ are means in an interval I, play an essential role in some problems connected with iterations of means. For

$$M(x,y) := \frac{x+y}{2} \quad \text{and} \quad N(x,y) := \sqrt{xy},$$

this equation appears in connection with the *AGM* iteration of Gauss and elliptic integrals (cf. for instance [1]).

Applying our Proposition 1 we can prove the following

Theorem 1. *Let $p > 0$ be arbitrarily fixed. Suppose that $f: (0,\infty)^2 \to \mathbb{R}$ is continuous on the diagonal $\Delta := \{(x,x): x > 0\}$. Then f satisfies the functional equation*

$$f\left(xy\sqrt{\frac{p^2+1}{p^2x^2+y^2}}, \sqrt{\frac{p^2x^2+y^2}{p^2+1}} \right) = f(x,y), \quad x,y > 0, \qquad (1)$$

if, and only if,

$$f(x,y) = F(xy), \quad x,y > 0,$$

where $F: (0,\infty) \to \mathbb{R}$ is a continuous function of a single variable.

Proof. Suppose that $f: (0,\infty) \times (0,\infty) \to \mathbb{R}$ is continuous and satisfies equation (1). Since

$$E_p(x,y) = xy\sqrt{\frac{p^2+1}{p^2x^2+y^2}}, \qquad M_{2,w^*}^{[2]}(x,y) = xy\sqrt{\frac{p^2x^2+y^2}{p^2+1}}, \quad x,y > 0,$$

we can write equation (1) in the form

$$f \circ [(E_p, M_{2,w^*}^{[2]})] = f.$$

Hence, by induction, we get

$$f \circ [(E_p, M_{2,w^*}^{[2]})^n] = f, \quad n \in \mathbb{N},$$

where $(E_p, M_{2,w^*}^{[2]})^n$ denotes the n-th iteration of the mean-type mapping $(E_p, M_{2,w^*}^{[2]})$. By Proposition 1, letting here $n \to \infty$ and making use of the continuity of f on Δ, we obtain

$$f \circ [(G_2, G_2)] = f,$$

that is

$$f(x, y) = f[(\sqrt{xy}, \sqrt{xy})], \quad x, y > 0.$$

Setting

$$F(u) := f(\sqrt{u}, \sqrt{u}), \quad u > 0,$$

we have

$$f(x, y) = F(xy), \quad x, y > 0.$$

Since the converse implication requires only simple calculations, the proof is complete.   □

Similarly, applying Proposition 2, we can prove

Theorem 2. *Let $p, q > 0$ be fixed. Suppose that $f: (0, \infty)^3 \to \mathbb{R}$ is continuous on the diagonal $\Delta := \{(x, x, x): x > 0\}$. Then f satisfies the functional equation*

$$f \circ (E_{p,q}(x, y, z), K_{p,q}(x, y, z), M_{3,w}^{[2]}{}_*(x, y, z)) = f(x, y, z), \quad x, y, z > 0,$$

if, and only if,

$$f(x, y, z) = F(xyz), \quad x, y, z > 0,$$

where $F: (0, \infty) \to \mathbb{R}$ is a continuous function of a single variable.

Remark 4. *A k-dimensional counterpart of the above results is also true. Its version for the symmetric means results from Proposition* 3.

Acknowledgement

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