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# The Functional Equation of Homogeneity and its Stability Properties

By

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## 1. Introduction

In [4] Z. Kominek and J. Matkowski investigated the stability behaviour of the functional equation

$$f(\alpha x) = \alpha f(x).$$

Contrasting earlier "superstability" results of Jósef Tabor ([6]) and of Jacek and Jósef Tabor ([7]) [4] contains *stability* results. This means that for approximately homogeneous functions the existence of homogeneous functions close to but different from the given ones is proved. The theorem reads as follows.

**Theorem 1.** ([4]) *Let $X$ be a real vector space and let $S \subseteq X$ be a cone* (i.e., $S \neq \emptyset$ and $\alpha S \subseteq S$ for all $\alpha > 0$). *Let $Y$ be a sequentially complete topological vector space which is Hausdorff. Then: If $f: S \to Y$ is such that*

$$\alpha^{-1} f(\alpha x) - f(x) \in V \quad (x \in S, \alpha \in A) \tag{1}$$

*where $V \subseteq Y$ is a bounded subset of $Y$ and $A$ is a subset of $]1, \infty[$ with nonempty interior, then there is a (unique) $F: S \to Y$* such that $F$ is *$\mathbb{R}_+$-homogeneous and such that $f - F$ is bounded* (by some bound depending on $V$).

Tabor's result ([6]) adapted to the above situation is the following.

*If $\alpha^{-1} f(\alpha x) - f(x) \in V \quad (x \in S, \alpha > 0)$ then $f$ is homogeneous* ("Superstability").

My aim is to generalize the results of Theorem 1. To motivate this generalization note that (1) may be rewritten as

$$f(\alpha x) - \alpha f(x) \in \alpha V \quad (x \in S, \alpha \in A), \tag{1'}$$

where $\alpha \mapsto \alpha$ is a (very special) multiplicative function and $\alpha V$ is bounded. Moreover multiplying elements of $S$ by positive reals is a special case of a *group action*. (The latter aspect also is taken into consideration in [7].)

Facts and notations in connection with topological vector spaces (*tvs*) are taken from [5]. A subset $B$ of a real or complex tvs $Y$ is called *bounded* if for every neighbourhood $U$ of $0 \in Y$ there is some scalar $\alpha > 0$ such that $B \subseteq \beta U$ for all $|\beta| \geq \alpha$. Obviously subsets of bounded sets and finite unions of bounded sets are bounded. Moreover $\lambda A + B$ is bounded if $A$ and $B$ are bounded and if $\lambda \in \mathbb{K}$ is arbitrary. ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ denotes the ground field of the tvs $Y$.) If $Y$ is locally convex then the absolute convex hull (and the convex hull) of a subset is also bounded. (Following the notation in [5] the convex hull and the absolute convex hull of a subset $A$ of $Y$ are denoted by $\mathrm{cx}(A)$ and $\mathrm{acx}\,(A)$ respectively.

The general setting used in the sequel is the following.

- $X \neq \emptyset$, a set,
- $G$ a semigroup (with or without unit),
- $\cdot : G \times X \to X$ a semigroup action of $G$ on $X$, i.e., $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g$, $h \in G$ and all $x \in X$, $1 \cdot x = x$ for all $x \in X$, if $G$ is a semigroup with unit 1,
- $Y$ a $\mathbb{K}$-tvs which is Hausdorff,
- $V : G \to \mathfrak{B}(Y)$ a mapping from $G$ into the set of $\mathfrak{B}(Y)$ of bounded subsets of $Y$,
- $f : X \to Y$,
- $M : G \to \mathbb{K}$.

## 2. Global Stability Results

We have the following

**Theorem 2.** *If*

$$f(\alpha x) - M(\alpha) f(x) \in V(\alpha) \quad (x \in X, \alpha \in G) \tag{2}$$

*and if $f$ is unbounded* (i.e., the set $f(X)$ is not bounded) *then we have*

$$M(\alpha\beta) = M(\alpha) M(\beta) \quad (\alpha, \beta \in G), \tag{3}$$

*i.e., $M : G \to \mathbb{K}$ is a "multiplicative" function.*

*Proof*: For $\alpha, \beta \in G$ and $x \in X$ we have—using (2)—

$$
\begin{aligned}
(M(\alpha\beta) - M(\alpha)M(\beta))f(x) = M(\alpha\beta)f(x) - f(\alpha\beta x) + f(\alpha(\beta x)) \\
- M(\alpha)f(\beta x) + M(\alpha)(f(\beta x)) \\
- M(\beta)f(x)) \in - V(\alpha\beta) \\
+ V(\alpha) + M(\alpha)V(\beta).
\end{aligned}
$$

Thus

$$
(M(\alpha\beta) - M(\alpha)M(\beta))f(X) \subseteq V_{\alpha,\beta} := - V(\alpha\beta)
$$

$$
+ V(\alpha) + M(\alpha)V(\beta) \in \mathfrak{B}(Y)
$$

implying by the unboundedness of $f$ that $(M(\alpha\beta) - M(\alpha\beta)) = 0$. □

**Corollary 1.** (Baker, Ger [1], [2].) *Let $M: G \to \mathbb{K}$ be such, that*

$$
|M(xy) - M(x)M(y)| \leq v(x) \quad (x, y \in G)
$$

*for some $v: G \to \mathbb{R}$. Then $M$ is either bounded or multiplicative.*

*Proof*: Take $X = G$, $Y = \mathbb{K}$, $f = M$, and $V(x) := \{y \in \mathbb{K} \mid |y| \leq v(x)\}$ and apply Theorem 2. □

According to Theorem 2 there is no loss of generality in assuming $M$ to be multiplicative provided that $f$ is unbounded. (The latter case seems to be the only interesting one, since for bounded $f$ the left-hand side of (2) is bounded with respect to $x \in X$ anyway.) Thus, from now on, we assume the multiplicativity of $M$; and we call $F: X \to Y$ "$M$-homogeneous" if

$$
F(\alpha x) = M(\alpha)F(x) \quad (x \in X, \alpha \in G).
$$

**Theorem 3.** *Let $G$ be a semigroup with unit, let $X$ and $Y$ be as above and assume that $Y$ is locally convex and sequentially complete. Suppose furthermore that for some $\alpha_0 \in G$ we have*

$$
f(\alpha\alpha_0 x) = f(\alpha_0\alpha x) \quad (x \in X, \alpha \in G) \quad \text{and} \quad |M(\alpha_0)| > 1. \tag{4}
$$

*Then condition (2) implies that there is a unique $M$-homogeneous function $F: X \to Y$ such that the difference $f - F$ is bounded. A bound is given by the set*

$$
(|M(\alpha_0)| - 1)^{-1} \operatorname{seqcl}(\operatorname{acx}(V(\alpha_0))), \text{ i.e.,}
$$

$$
(f - F)(X) \subseteq \frac{1}{|M(\alpha_0)| - 1} \operatorname{seqcl}(\operatorname{acx}(V(\alpha_0))). \tag{5}
$$

*Moreover, if* $\mathbb{K}=\mathbb{R}$ *and if* $M(\alpha_0)>1$

$$(f-F)(X)\subseteq\frac{1}{M(\alpha_0)-1}\operatorname{seqcl}(\operatorname{cx}(V(\alpha_0)\cup\{0\})). \tag{6}$$

(*The sequential closure of* $V\subseteq Y$ *is denoted by* $\operatorname{seqcl}(V)$.)

*Proof*: Suppose that $F$ is $M$-homogeneous and assume that the image of $X$ under $f-F$ is contained in $W\in\mathfrak{B}(Y)$:

$$(f-F)(X)\subseteq W.$$

Then

$$f(\alpha_0^n x)-F(\alpha_0^n x)=f(\alpha_0^n x)-M(\alpha_0)^n F(x)\in W \quad (x\in X, n\in\mathbb{N})$$

or

$$M(\alpha_0)^{-n}f(\alpha_0^n x)-F(x)\in M(\alpha_0)^{-n}W$$

implying – since $W$ is bounded – that

$$F(x)=\lim_{n\to\infty}\frac{f(\alpha_0^n)}{M(\alpha_0)^n}. \tag{7}$$

This shows uniqueness.

To show existence (6) gives a hint what to do. Let us define

$$\varphi_n(x):=\frac{f(\alpha_0^n x)}{M(\alpha_0)^n}.$$

Then

$$\varphi_{n+1}(x)-\varphi_n(x)=\frac{1}{M(\alpha_0)^{n+1}}(f(\alpha_0^{n+1}x)-M(\alpha_0)f(\alpha_0^n x))\in\frac{1}{M(\alpha_0)^{n+1}}V(\alpha_0).$$

Putting $k:=M(\alpha_0)$ and taking $\ell>0$ we get for all $n\in\mathbb{N}_0$

$$\begin{aligned}\varphi_{n+\ell}(x)-\varphi_n(x)&=\sum_{j=n}^{n+\ell-1}(\varphi_{j+1}(x)-\varphi_j(x))\in\sum_{j=n}^{n+\ell-1}k^{-(j+1)}V(\alpha_0)\\&\subseteq\sum_{j=n}^{n+\ell-1}|k|^{-(j+1)}\operatorname{acx}(V(\alpha_0))\\&\subseteq\frac{1}{|k|^n(|k|-1)}\operatorname{acx}(V(\alpha_0)).\end{aligned} \tag{8}$$

The latter inclusion is valid since for absolute convex set $W$ the relations

$$rW+sW\subseteq(|r|+|s|)W \quad \text{and} \quad pW\subseteq qW \quad (p,q,r,s\in\mathbb{K},|p|\leq|q|)$$

hold true.

Since $\operatorname{acx}(V(\alpha_0))$ is also bounded (8) shows that the sequence $(\varphi_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence and that

$$\varphi_n(x) - \varphi_0(x) \in \frac{1}{|k|-1} \operatorname{acx} V(\alpha_0). \tag{9}$$

Since $Y$ is sequentially complete the pointwise limit $F := \lim_{n\to\infty} \varphi_n$ exists. Letting $n$ tend to $\infty$ in (9) and noting that $\varphi_0 = f$ then leads to

$$F(x) - f(x) \in \frac{1}{|k|-1} \operatorname{seqcl}(\operatorname{acx}(V(\alpha_0))) \quad (x \in X). \tag{10}$$

Now we show that $F$ is $M$-homogeneous. We consider the difference

$$\varphi_n(\alpha x) - M(\alpha)\varphi_n(x)$$

and get by using the first part of (4)

$$\begin{aligned}\varphi_n(\alpha x) - M(\alpha)\varphi_n(x) &= M(\alpha_0)^{-n}(f(\alpha_0^n \alpha x) - M(\alpha) f(\alpha_0^n x))\\ &= M(\alpha_0)^{-n}(f(\alpha \alpha_0^n x) - M(\alpha) f(\alpha_0^n x))\\ &\in M(\alpha_0)^{-n} V(\alpha)\end{aligned}$$

This shows that $\varphi_n(\alpha x) - M(\alpha)\varphi_n(x)$ tends to 0 when $n$ goes to $\infty$. But this means that $F(\alpha x) - M(\alpha x) F(x) = 0$ (for all $\alpha \in G$ and all $x \in X$) as desired.

Similar arguments may be used when $\mathbb{K} = \mathbb{R}$ and $k := M(\alpha_0) > 1$. Denoting by $V'$ the convex closure of $V(\alpha_0) \cup \{0\}$ and observing that for convex sets $W$ and real numbers $r_1, \dots, r_m > 0$

$$r_1 W + r_2 W + \cdots + r_m W \subseteq \left(\sum_{j=1}^{m} r_j\right) W$$

and that moreover

$$rW \subseteq sW \quad \text{if} \quad 0 \in W \quad \text{and} \quad 0 \le r \le s$$

we get

$$\varphi_{n+\ell}(x) - \varphi_n(x) \in \frac{1}{k^n(k-1)} V'. \tag{8'}$$

From this point on one may argue as before. □

**Remark 1.** The condition $f(\alpha\alpha_0 x) = f(\alpha_0 \alpha x)$ for all $\alpha \in G$ and all $x \in X$ holds for example when $\alpha_0 \in C(G)$, the center of $G$; in particular it holds for abelian semigroups $G$.

**Remark 2.** The theorem becomes false if $M = 1$. A counterexample is provided by

$$G = X = Y = \mathbb{R}, \alpha \cdot x := \alpha + x, f(x) := \int_0^x \frac{1}{\sqrt{1+|t|}}\, dt, V(\alpha)$$
$$:= \{y \in \mathbb{R} \mid |y| \leq |\alpha|\},$$

because $f$ is unbounded ($f(x) = 2\,\mathrm{sign}(x)(\sqrt{1+|x|} - 1)$ and because the only 1-homogeneous functions are the constant ones.

## 3. Local Results

Theorem 3 does not cover the situation given in Theorem 1 since (1′) is supposed to hold for $\alpha \in A$ only. (Kominek-Matkowski consider the situation when $X = S$ is a cone in some real vector space, $G = \mathbb{R}_{>0}$, the set of all positive reals, and $M(\alpha) = \alpha$.) But note that the condition $A^\circ \neq \emptyset$ used in [4] implies that $A$ generates the multiplicative group $\mathbb{R}_{>0}$: $\mathbb{R}_{>0} = \langle A \rangle$.

The following theorem covers Theorem. 1.

**Theorem 4.** *Let G be a semigroup with unit acting on the non-empty set X. Let $A \subseteq G$ generate G as a semigroup, i.e.*

$$G = \langle A \rangle_s := \text{set of all finite products of elements in } A.$$

*Suppose that $f: X \to Y$, $Y$ a locally convex sequentially complete tvs over $\mathbb{K}$, and $M: G \to \mathbb{K}$, $M$ multiplicative, satisfy*

$$f(\alpha x) - M(\alpha) f(x) \in V(\alpha) \quad (\alpha \in A, x \in X), \tag{2′}$$

*where $V: A \to \mathfrak{B}(Y)$.*

*Then there exists a (unique) M-homogeneous function $F: X \to Y$ such that $f - F$ is bounded provided that for some $\alpha_0 \in A$ we have*

$$f(\alpha\alpha_0 x) = f(\alpha_0 \alpha x) \quad (\alpha \in A, x \in X) \quad \text{and} \quad |M(\alpha_0)| > 1.$$

*If G is a group and if A generates G as a group, $G = \langle A \rangle$, the assertion of the theorem remains true provided that we assume the same hypotheses as before.*

*Proof*: Obviously it is enough to show the existence of some $V': G \to \mathfrak{B}(Y)$ such that

$$f(\alpha x) - M(\alpha) f(x) \in V'(\alpha) \quad (\alpha \in G, x \in X)$$

holds, since then we may apply Theorem 3.

For $n \in \mathbb{N}_0$ let $A_n$ be the set of products $\Pi_{j=1}^{\ell} \alpha_j$ with $0 \leq \ell \leq n$ and arbitrary $\alpha_1, \ldots, \alpha_\ell \in A$. Define

$$V_0 : A_0 \to \mathfrak{B}(Y)$$

by $V_0(1) := \{0\}$; then

$$f(\alpha x) - M(\alpha) f(x) \in V_\ell(\alpha) \quad (\alpha \in A_\ell, x \in X) \tag{11}$$

for $\ell = 0$ since $M(1) = 1$. Now, suppose that (11) holds for some $\ell \geq 0$ where

$$V_\ell : A_\ell \to \mathfrak{B}(Y)$$

is such that the restriction $V_\ell | A_{\ell-1}$ of $V_\ell$ to $A_{\ell-1}$ satisfies $V_\ell | A_{\ell-1} = V_{\ell-1}$. Consider $\gamma \in A_{\ell+1} \setminus A_\ell$ and write $\gamma = \alpha\beta$ with $\alpha \in A_\ell$, $\beta \in A$. Then

$$\begin{aligned} f(\gamma x) - M(\gamma) f(x) &= f(\alpha\beta x) - M(\alpha) f(\beta x) + M(\alpha)\left(f(\beta x) - M(\beta) f(x)\right) \\ &\in V_\ell(\alpha) + M(\alpha)\, V(\beta) =: V_{\ell+1}(\gamma) \end{aligned}$$

showing how to define $V_{\ell+1}$.

The function $V'$, defined by $V' | A_\ell := V_\ell$ for all $\ell$, then has the required properties, at least in the semigroup case.

In the group case we observe that $G$ is generated by $A \cup A^{-1}$ as a semigroup. Thus it is enough to find a suitable extension of $V$ to $A \cup A^{-1}$ (and to use the foregoing procedure). Note that $M(\alpha) \neq 0$ for all $\alpha \in G$ since $G$ is a group. So let us define

$$V(\alpha^{-1}) := -M(\alpha)^{-1} V(\alpha) \quad (\alpha \in A \setminus A^{-1}).$$

Then $f(\alpha x) - M(\alpha) f(x) \in V(\alpha)$ for $\alpha^{-1} x$ instead of $x$ implies

$$f(x) - M(\alpha) f(\alpha^{-1} x) \in V(\alpha) \quad \text{or} \quad f(\alpha^{-1} x) - M(\alpha^{-1}) f(x) \in V(\alpha^{-1}). \ \square$$

**Remark 3.** Even in the Kominek-Matkowski case the condition that $A$ generates $G$ as a group is weaker than the weakest condition given in [4]. This condition reads as

$$A \subseteq ]1, \infty[ \quad \text{and} \quad (\underbrace{A \cdot \ldots \cdot A}_{n \text{ times}})^\circ \neq \varnothing \quad \text{for some} \quad n \in \mathbb{N}.$$

In fact, there are subsets $A$ of $]1, \infty[$ such that

$$\langle A \rangle = \mathbb{R}_{>0} \quad \text{and} \quad (\underbrace{A \cdot \ldots \cdot A}_{n \text{ times}})^\circ = \varnothing \quad \text{for all} \quad n \in \mathbb{N}.$$

An example is given by the following. Let $H \subseteq [0, \infty[$ be a Hamel basis of $\mathbb{R}$ and put $A' := \bigcup_{n \in \mathbb{N}} \frac{1}{n!} H$. Then it is easy to see that $A'$ generates the additive group $\mathbb{R}$ and that $(\underbrace{A' + \cdots + A'}_{n \text{ times}})^\circ = \varnothing$ for all $n \in \mathbb{N}$. Thus $A := \exp(A')$ has the desired property.

R. Ger (personal communication) points out that also $\exp(A'')$ with

$$A'' = \sum_{h \in H} \mathbb{Q}_{\geq 0} h$$

has the desired property. ($A''$ is the set of finite linear combinations of elements of $H$ with non negative rational coefficients.)

## 4. A Noncommutative Example

Of course there are many special cases of the above theorems, among others those given in [4]. Here I want to present an example where the (semi-)group involved is not abelian.

**Theorem 5.** *Let $X$ be a finite dimensional $\mathbb{K}$-vector space of dimension $n$ greater than 1 and let the group $G := \mathrm{Aut}(X)$ of linear automorphisms of $X$ act on $X$ in the canonical way ($Ax = A(x)$). Let furthermore $M : G \to \mathbb{K}$ be multiplicative such that $|M| \neq 1$. Then, given any mapping $V : G \to \mathfrak{B}(Y)$ where $Y$ is a locally convex and Hausdorff tvs over $\mathbb{K}$ (not necessarily sequentially complete), we have that any $f : X \to Y$ such that*

$$f(Ax) - M(A) f(x) \in V(A) \quad (A \in G, x \in X)$$

*is bounded; especially this means that the zero function is the only $M$-homogeneous function mapping $X$ into $\mathbb{K}$.*

*Proof*: Due to [3] the multiplicative function $M$ may be represented in the form

$$M(A) = m(\det(A)) \quad \text{with} \quad m : \mathbb{K}^* \to \mathbb{K} \quad \text{multiplicative.}$$

If $M = 0$ the assertion is obviously true ($f(\mathrm{id}_X x) \in V(\mathrm{id}_X)$). Otherwise $M(G) \subseteq \mathbb{K}^* := \mathbb{K} \setminus \{0\}$. By assumption we have $|M| \neq 1$. Thus there is some $A_1 \in G$ such that $|M(A_1)| > 1$. Our next aim is to show that $F = 0$ is the only $M$-homogeneous function.

Let $F : X \to Y$ be $M$-homogeneous. Then $F(0) = F(A_1 0) = M(A_1) F(0)$; thus $F(0) = 0$. If $0 \neq x \in X$, then there is a basis $\{x_1 = x, x_2, \ldots, x_n\}$ of $X$ containing $x$. For $\lambda_2, \lambda_3, \ldots, \lambda_n \in \mathbb{K}^*$ the set $\{y_1 = x, y_2 = \lambda_2 x_2, \ldots, y_n = \lambda_n x_n\}$ is another basis of $X$. Then there is some $A$ in $G$ such that $A(x_j) = y_j$ for all $1 \leq j \leq n$. In particular we have $A(x) = x$. Choosing $\lambda_j := 1, 2 \leq j \leq n-1$ and $\lambda_n := \det(A_1)$ implies $M(A) = m(\det(A)) = m(\det(A_1)) = M(A_1)$ since $\det(A) = \lambda_2 \cdot \lambda_3 \cdot \ldots \cdot \lambda_n$. Thus $F(x) = F(Ax) = M(A) F(x) = M(A_1) F(x)$ and $F(x) = 0$, as desired.

To finish the proof we may assume that $Y$ is also sequentially complete, because we may embed $Y$ into a (sequentially) complete locally convex and Hausdorff tvs ([J]) and because subsets of $Y$ are bounded in the completion iff they are bounded in $Y$. Furthermore we may assume that $\lambda := \det(A_1)$ has an $n$th root in $\mathbb{K}$. This is trivial for $\mathbb{K} = \mathbb{C}$. If $\mathbb{K} = \mathbb{R}$ this can be achieved by using $A_1^2$ instead of $A_1$ which makes $\lambda$ positive. In any case $A_0 := \sqrt[n]{\lambda}\,\mathrm{id}_X$ lies in the center of $G$. Moreover

$$|M(A_0)| = |m(\sqrt[n]{\lambda^n})| = |M(A_1)| > 1.$$

□

## 5. Tabor's Superstability Result

The following theorem generalizes some results of [7].

**Theorem 6.** *Let $X$ and $G$ be as in the introduction, $G$ with unit, let $Y$ be a Hausdorff tvs and $Y_1$ a subset of $Y$ such that $G$ operates on $Y_1$ with $y \mapsto \alpha y$ continuous for all $\alpha \in G$, and let $V: G \times X \to \mathfrak{B}(Y)$ be such, that there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of invertible elements in $G$ with the property that for arbitrary $\alpha \in G$ and $x \in X$*

$$\lim_{n\to\infty} y_n = 0$$

*for all sequences $(y_n)$ where $y_n \in V(\alpha\alpha_n, \alpha_n^{-1}x)$. Then any function $f: X \to Y_1$ satisfying*

$$f(\alpha x) - \alpha f(x) \in V(\alpha, x) \quad (\alpha \in G, x \in X) \tag{12}$$

*is homogeneous, i.e.*

$$f(\alpha x) = \alpha f(x) \quad (\alpha \in G, x \in X).$$

*(The operation of $G$ on $Y_1$ is again denoted by $(\alpha, y) \mapsto \alpha y$.)*

*Proof*: Using (12) for $\alpha\alpha_n$ and $\alpha_n^{-1}x$ gives

$$f(\alpha x) - \alpha\alpha_n f(\alpha_n^{-1}x) \in V(\alpha\alpha_n, \alpha_n^{-1}x) \quad (\alpha \in G, x \in X, n \in \mathbb{N})$$

showing that

$$f(\alpha x) = \lim_{n\to\infty} \alpha\alpha_n f(\alpha_n^{-1}x) \quad (\alpha \in G, x \in X).$$

Putting $\alpha = 1$ and using the continuity of $y \to \alpha y$ thus yields

$$f(x) = \lim_{n\to\infty} \alpha_n f(\alpha_n^{-1}x) \quad (x \in X)$$

and

$$f(\alpha x) = \lim_{n\to\infty} \alpha\alpha_n f(\alpha_n^{-1}x) = \alpha \lim_{n\to\infty} \alpha_n f(\alpha_n^{-1}x) = \alpha f(x) \quad (\alpha \in G, x \in X).$$

□

**Remark 4.** The original result was the special case that $\mathbb{R}$ operates on invariant subsets $X$ of a real vectorspace and $Y_1$ of a Hausdorff tvs by multiplication by scalars. $V(\alpha,x)$ was of the form $g(\alpha,x)V$ with $V \in \mathfrak{B}$ and $g: G \times X \to \mathbb{R}$ satisfying $\lim_{n\to\infty}(\alpha\alpha_n, \alpha_n^{-1}x) = 0$. It is easy to see that the bounded sets $g(\alpha,x)V$ are a (very) special case of sets $V(\alpha,x)$ as above.

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