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# Analysis of Photon Shapes and Prediction of a New Quantization Rule

By

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## Abstract

The Schrödinger equation is solved for spontaneous and stimulated photon emission out of an ideal energy well, a box inescapable for electrons, which can change their state only by odd quantum numbers. Shapes of photons emanated spontaneously and under stimulation are calculated and differ from each other. Calculations of the line widths of spontaneously emitted photons are in agreement with current theories and turn out to be much smaller than those of photons emitted by stimulation. The latter exhibit a line width proportional to the square root of intensity (power). The absorption of the stimulating electromagnetic wave shows a saturation effect, explaining a well known phenomenon. For very short photons, the basic energy relation $W = \hbar\omega_{21}$ becomes questionable, corroborating a recent result obtained by Keller, although possible to obey, if a quantization rule for the time duration of the photon is introduced.

## I. Introduction

The problem under investigation was raised by the recent claim [1] of a "fundamental discovery", the frequency limitation, which is supposed to be caused by a detector's need of "at least one quantum $\hbar\omega$ and with $\omega \to \infty$ the signal's energy would become infinite". The claim is based on the belief that the high frequency part of the Fourier spectrum of a wave packet, say of a photon, has something to do with the energy quantum $\hbar\omega$ of the whole packet. The problem is related to the question as to whether a photon starts and stops at certain times to be emanated, and what shapes photons may acquire. Since photons are in many cases emanated by an electron falling from an energy level to a lower one, where both levels correspond to steady states, there should definitely be a certain length of the wave packet. But how long is it and what determines its shape? Is there a difference between spontaneously emitted photons and photons emitted through stimulation? The space-time description of photon emission is usually dealt with by quantum-electrodynamics. The state of the art is presented in a recent article of Keller [2] who found for photon emission from an atom that for very short pulse trains "pronounced deviations from the textbook result, energy $W = \hbar\omega$, occur".

In the present article a simple and transparent model of photon emission from electrons captured in an energy well is used to answer the question of the "frequency limitation", to check the validity of the energy relation $W = \hbar\omega$ particularly for short photons and find how to possibly save it by a proper rule. Solutions of Schrödinger's fundamental wave equation a priori disregarding spin and magnetic forces (a typical engineering theory, as some critical readers of the manuscript maintained) suffice to answer the questions.

## II. Radiation of a Photon from an Excited Electron in an Ideal Energy Well

Since a nonlinear problem is to be solved, the author prefers to write Schrödinger's equation in real notation for the wave functions $\Psi$ and $\Phi$ instead of the conventional complex functions $\psi$ and $\psi^*$. With

$$\psi = \Psi + i\Phi, \tag{1}$$

the wave equations read

$$\Delta\Psi - \frac{2m}{\hbar}\frac{\partial\Phi}{\partial t} = \frac{2m}{\hbar^2}W\Psi, \tag{2}$$

$$\Delta\Phi + \frac{2m}{\hbar}\frac{\partial\Psi}{\partial t} = \frac{2m}{\hbar^2}W\Phi. \tag{3}$$

Here $m$ is the electron mass, $\hbar$ Planck's constant $h$ divided by $2\pi$, $W$ the potential energy within the well (actually a box) situated at $0 < x < d_x, 0 < y < d_y, 0 < z < d_z$, and $\Delta$ is Laplace's operator.

With a flat bottom of the well in the steady state, insurmountable walls and an electric field in $x$-direction, whose origin is internal (from the emanating photon) and maybe also external,

$$E_x = \hat{E}(t)\sin(\omega_2 - \omega_1)t. \tag{4}$$

For spontaneous emission, the internal field prevails, for stimulated emission the external field dominates. The potential energy, with the reference potential in the well's center place, becomes

$$W = -e\hat{E}\left(\frac{d_x}{2} - x\right)\sin(\omega_2 - \omega_1)t, \tag{5}$$

which implies that the well is small compared to $1/4$ of the electromagnetic wavelength. Here $\hbar\omega_2$ is the initial and $\hbar\omega_1$ the final energy of the electron of charge $-e$. The amplitude $\hat{E}$ of the (within the well) space-independent electric field may be time-modulated. The space-dependent factor $d_x/2 - x$ in Eq. (5) is for a purpose evident from the following analysis replaced by a symmetric Fourier series of period $2d_x$, so that

$$W = -4ed_x\hat{E}\sin(\omega_2 - \omega_1)t\sum_{k=1}^{\infty}\frac{\cos(\pi(2k-1)x/d_x)}{(2k-1)^2\pi^2}. \tag{6}$$

The difference between the functions in Eqs. (5) and (6) lies in the fact that according to Eq. (5), the electric field is constant including the walls, whereas according to Eq. (6), the electrical field is constant within the well but drops to zero at the potential walls by step functions. There is no consequence since both wave functions are zero at the walls, and moreover, the superimposed trapping field is infinity on the wall. The alternative antisymmetric Fourier series which could replace the linear function $d_x/2 - x$ within the well does not lead to a solution and had therefore been disregarded.

Now an electron at a steady-state energy level of

$$\hbar\omega_2 = \frac{\hbar^2}{2m}\left[\left(\frac{l\pi}{d_x}\right)^2 + \left(\frac{g\pi}{d_y}\right)^2 + \left(\frac{n\pi}{d_z}\right)^2\right] \tag{7}$$

is considered to fall back to the empty steady-state level

$$\hbar\omega_1 = \frac{\hbar^2}{2m}\left[\left(\frac{(l-l_0)\pi}{d_x}\right)^2 + \left(\frac{g\pi}{d_y}\right)^2 + \left(\frac{n\pi}{d_z}\right)^2\right] \tag{8}$$

with $l$, $l-l_0$, $g$, and $n$ as quantum numbers. The dynamics of this process may be described by the wave functions

$$\Psi\left(\frac{d_x d_y d_z}{8}\right)^{1/2} = \left[A(t)\sin\omega_2 t\cdot\sin\frac{l\pi x}{d_x} + B(t)\sin\omega_1 t \cdot\sin\frac{(l-l_0)\pi x}{d_x}\right]\sin\frac{g\pi y}{d_y}\cdot\sin\frac{n\pi z}{d_z}, \qquad (9)$$

$$\Phi\left(\frac{d_x d_y d_z}{8}\right)^{1/2} = \left[A(t)\cos\omega_2 t\cdot\sin\frac{l\pi x}{d_x} + B(t)\cos\omega_1 t \cdot\sin\frac{(l-l_0)\pi x}{d_x}\right]\sin\frac{g\pi y}{d_y}\cdot\sin\frac{n\pi z}{d_z}, \qquad (10)$$

which imply constant frequencies within the envelopes. The amplitudes $A(t)$ and $B(t)$ are normalized such as to equal unity in the steady states. In the dynamic case with modulated amplitudes, it is expected that $A^2$ starts from unity and terminates at zero, while $B^2$ starts at zero and terminates at unity. Since the electron is supposed to be partitioned between the levels $\hbar\omega_2$ and $\hbar\omega_1$, it is expected and proven below that

$$A^2(t) + B^2(t) = 1. \qquad (11)$$

With Eqs. (9) and (10) introduced into Schrödinger's equations (2) and (3), in the latter's right hand sides only the resonant terms are considered, i. e., the terms with the same space and time dependences as appear on the left hand sides. It can immediately be recognized that in the Fourier series of Eq. (6) only the terms $2k_1 - 1 = l_0$ and $2k_2 - 1 = 2l - l_0$ matter. Thus $l_0$ has to be an odd number, which will be shown later to be in agreement with current theories. Furthermore, the degenerate case $\omega_2 = 2\omega_1$ must be excluded. With these restrictions, Eqs. (9), (10), (2), and (3) lead to

$$\frac{dA}{dt}\cos\omega_2 t\cdot\sin\frac{l\pi x}{d_x} + \frac{dB}{dt}\cos\omega_1 t\cdot\sin\frac{(l-l_0)\pi x}{d_x} = \alpha(t)\left[A\cos\omega_1 t\cdot\sin\frac{(l-l_0)\pi x}{d_x} - B\cos\omega_2 t\cdot\sin\frac{l\pi x}{d_x}\right] \qquad (12)$$

and

$$\frac{dA}{dt}\sin\omega_2 t\cdot\sin\frac{l\pi x}{d_x} + \frac{dB}{dt}\sin\omega_1 t\cdot\sin\frac{(l-l_0)\pi x}{d_x} = \alpha(t)\left[A\sin\omega_1 t\cdot\sin\frac{(l-l_0)\pi x}{d_x} - B\sin\omega_2 t\cdot\sin\frac{l\pi x}{d_x}\right], \qquad (13)$$

respectively, with the abbreviation

$$\alpha(t) = \frac{4ed_x\hat{E}(t)}{\pi^2\hbar}\frac{l(l-l_0)}{l_0^2(2l-l_0)^2}. \tag{14}$$

These equations can be met if

$$\frac{dA}{dt} = -\alpha(t)B, \tag{15}$$

$$\frac{dB}{dt} = \alpha(t)A, \tag{16}$$

which allows two general conclusions. The first is immediately evident if we multiply Eq. (15) with $A$, Eq. (16) with $B$, and integrate their sum. This results in Eq. (11) and confirms the normalization criterion.

To reach the second conclusion, the electron current density has to be considered first. From the general equation

$$\vec{S} = \frac{e\hbar}{m}\left(\Phi\vec{\nabla}\Psi - \Psi\vec{\nabla}\Phi\right) \tag{17}$$

and Eqs. (9) and (10), the current density reads

$$S_x = -\frac{8e\hbar\pi}{md_x^2d_yd_z}AB\sin\omega_{21}t\cdot\sin^2\frac{g\pi y}{d_y}\cdot\sin^2\frac{n\pi z}{d_z} \times\left[\left(l-\frac{l_0}{2}\right)\sin\frac{l_0\pi x}{d_x}-\frac{l_0}{2}\sin\frac{(2l-l_0)\pi x}{d_x}\right], \tag{18}$$

with the abbreviation

$$\omega_{21} = \omega_2 - \omega_1. \tag{19}$$

The current effective in the radiation process (influence current) is derived from

$$I_{\text{infl}} = \frac{1}{d_x}\int_{x=0}^{d_x}\int_{y=0}^{d_y}\int_{z=0}^{d_z} S_x d_x d_y d_z, \tag{20}$$

which yields

$$I_{\text{infl}} = -\frac{8e\hbar}{md_x^2}AB\frac{l(l-l_0)}{l_0(2l-l_0)}\sin\omega_{21}t. \tag{21}$$

With Eqs. (4) and (14), the voltage $U = E_x d_x$ becomes

$$U = \frac{\pi^2 \hbar}{4e} \alpha(t) \frac{l_0^2 (2l - l_0)^2}{l(l - l_0)} \sin \omega_{21} t, \tag{22}$$

and the radiated power is calculated as

$$P = -\overline{I_{\text{infl}} U} = \frac{\pi^2 \hbar^2}{m d_x^2} l_0 (2l - l_0) \alpha(t) AB. \tag{23}$$

This result implies that $\alpha(t)AB$ varies slowly with time such as to justify the replacement of $\sin^2 \omega_{21} t$ in the averaging process by $1/2$. For "extremely short" photons, however, this is not the case and further considerations will be necessary.

From Eqs. (15) and (16), we have

$$\alpha(t) AB = -\frac{1}{2} \frac{dA^2}{dt}, \tag{24}$$

so that

$$\int_{-\infty}^{+\infty} \alpha AB dt = -\frac{A^2}{2} \Big|_{-\infty}^{+\infty} = \frac{1}{2} \tag{25}$$

Taking Eqs. (23), (24), and (7), and (8) together, we obtain the total energy of the photon

$$W = \int_{-\infty}^{+\infty} P dt = \frac{\hbar^2}{2m} \left( \frac{\pi}{d_x} \right)^2 l_0 (2l - l_0) = \hbar \omega_{21}, \tag{26}$$

with

$$\omega_{21} \approx 5.713 \cdot 10^{14} \mathrm{s}^{-1} \left( \frac{\mathrm{nm}}{d_x} \right)^2 l_0 (2l - l_0). \tag{27}$$

This result is not new but reassuring, since it states that regardless of whether the Fourier spectra show a long or short high-frequency tail, the energy formula $W = \hbar \omega_{21}$ is not violated. The basis holds, it is the frequency within the envelope which counts, and not parts of the Fourier spectrum. Extremely short photons may be an exception and will be dealt with in Section (III-B).

## III. Case Studies: Long and Short Photons, Spontaneous and Stimulated Emission

Obviously the photon is shaped by its environment. Suppose it may radiate freely without stimulation. Then the Hertzian dipole [3] appears as a proper model. Its radiation resistance offered to the current of amplitude $\hat{I}$ flowing through the dipole of length $d_x$ is, calculated for the radian frequency $\omega$,

$$R = \left(\frac{\omega d_x}{c}\right)^2 \frac{\sqrt{\mu/\varepsilon}}{6\pi}. \tag{28}$$

Here $c = (\mu\varepsilon)^{-1/2}$ is the velocity of light, $\varepsilon$ is the permittivity and $\mu$ the permeability of vacuum. The non-propagating electromagnetic energy stored in the near field, calculated outside a sphere of radius $d_x/2$, is time-averaged over rapid oscillations with $2\omega$, but allowing slow variations of $\hat{I}$,

$$\overline{W} = \frac{\mu\hat{I}^2}{2\pi}\left(\frac{d_x}{3} + \frac{2c^2}{3\omega^2 d_x}\right). \tag{29}$$

The corresponding energy of a small-band signal stored in a reactance $X(\omega)$ is

$$\overline{W} = \frac{\hat{I}^2}{4}\frac{dX}{d\omega}, \tag{30}$$

which is sometimes referred to as Foster's theorem [4]. From Eqs. (29) and (30), it can be seen that the near field may be represented by an inductance $L$ and capacitance $C$ in series with the radiation resistance, Eq. (28), with the values

$$L = \frac{2\mu d_x}{3\pi}, \quad C = \frac{3\pi\varepsilon d_x}{4}. \tag{31}$$

The fact that the energy stored in the near-field is represented by an inductance and a capacitance does not indicate a quasi-stationary approximation or an engineer's approach; it simply reflects the non-propagating (evanescent) nature of the near-field. Figure 1 depicts the current loop representing the radiation into free space. In Maxwell's

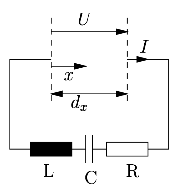


Fig. 1. Equivalent circuit for the interaction of the electrons inside a quantum box of extension $d_x$ with a dipole field; $R$ is the radiation resistance, the inductance $L$ and the capacitance $C$ represent the energy stored in the non-propagating near field

field theory, the total current density (electron-current density plus displacement-current density) equals curl$\vec{H}$ and therefore is source-free. Thus, with a quasi-one-dimensional electric field within the quantum box, averaging of the total current over $0 \leq x \leq d_x$ yields the influence current of Eq. (20) plus the capacitive current across the box,

$$I = I_{\text{infl}} + C_i \frac{dU}{dt}, \tag{32}$$

with

$$C_i = \frac{\varepsilon d_y d_z}{d_x} \approx \varepsilon d_x, \tag{33}$$

so that the voltage across the well becomes

$$-U = \frac{1}{C} \int_{-\infty}^{t} I dt + RI + L \frac{dI}{dt}. \tag{34}$$

Note that the application of the Hertzian model appears reasonable only for a cubic electron trap ($d_x = d_y = d_z$), of course, since otherwise the error caused by calculating the stored energy outside of the well using a spherical boundary would lead to unacceptable errors. Now we look out for solutions of Eqs. (15), (16), and (32)–(34).

### A. Low-frequency photon radiating spontaneously into free space

For negligible loss, the resonance frequency of the circuit shown in Fig. 1 would be given by the reactance zero of $L$, $C$, and $C_i$ in series,

$$\omega_0 = \frac{c}{d_x} \left( 2 + \frac{3\pi}{2} \right)^{1/2}. \tag{35}$$

While the radiation resistance, Eq. (28), would strongly dampen and prevent resonant behavior, it can still be argued that all frequencies $\omega$ far below $\omega_0$ are considered as "low". Since the model for the potential well implies constant electric field across $d_x$, which holds for

$$\frac{\omega_{21} d_x}{c} \ll \frac{\pi}{2}, \tag{36}$$

the radiation frequency has necessarily to be considered as low. With the time-varying amplitudes $\hat{I}_{\text{infl}}$ and $\hat{U}$ from Eqs. (21) and (22), the current from Eq. (32) reads

$$I = \left( -\hat{I}_{\text{infl}} + C_i \frac{d\hat{U}}{dt} \right) \sin \omega_{21} t + \omega_{21} C_i \hat{U} \cos_{21} t. \qquad (37)$$

Taking the the power balance of the circuit shown in Fig. 1 for a narrow linewidth photon one obtains

$$\frac{\hat{I}_{\text{infl}} \hat{U}}{2} = \left[ \left( -\hat{I}_{\text{infl}} + C_i \frac{d\hat{U}}{dt} \right)^2 + \left( \omega_{21} C_i \hat{U} \right)^2 \right] \frac{R}{2} + \frac{d\overline{W}}{dt}. \qquad (38)$$

The left hand side stands for the power delivered by the electron. On the right-hand side, the first term is the power dissipated in the sink, represented by the radiation resistance of Eq. (28). The second term considers the power fed into or delivered from the near-field energy.

It will be proven below, that the time constant $\tau$ associated with the time variations of $\hat{I}_{\text{infl}}$ and $\hat{U}$ exceeds all other time constants by many orders of magnitude, so that

$$\omega_{21} \tau \gg 1 \qquad (39)$$

and all time derivatives of the amplitudes in Eq. (38) can be neglected. Furthermore,

$$\frac{1}{R} \gg \omega_{21} C_i, \qquad (40)$$

so that Eq. (38) can be reduced to the simple condition

$$\frac{R \hat{I}_{\text{infl}}}{\hat{U}} = 1, \qquad (41)$$

which, under the conditions (39) and (40) for the exact determination of the near-field energy to become superfluous, should hold also for non-cubic traps. With Eq. (41), solutions of Eqs. (15), (16), (21) and (22) are readily found and read for $-\infty < t < \infty$

$$A = \frac{e^{-t/\tau}}{(1 + e^{-2t/\tau})^{1/2}}, \qquad (42)$$

$$B = \frac{1}{(1 + e^{-2t/\tau})^{1/2}}, \qquad (43)$$

$$\alpha = \frac{1}{2\tau} \frac{1}{\cosh t/\tau}. \qquad (44)$$

The time constant is calculated with the aid of Eqs. (21), (22), (28), and (41), which gives

$$\tau = \frac{\pi^2 m d_x^2}{32e^2}\frac{1}{R}\frac{l_0^3(2l-l_0)^3}{l^2(l-l_0)^2}$$
$$= \frac{3}{4\pi}\frac{m^3 d_x^4}{e^2\hbar^2\mu^{3/2}\varepsilon^{1/2}}\frac{l_0(2l-l_0)}{l^2(l-l_0)^2}$$
$$\approx 1.51\cdot 10^{-7}\mathrm{s}\left(\frac{d_x}{\mathrm{nm}}\right)^4\frac{l_0(2l-l_0)}{l^2(l-l_0)^2}. \qquad (45)$$

The condition (39) is easily met since from Eq. (27),

$$\omega_{21}\tau \approx 8.6\cdot 10^7\left(\frac{d_x}{\mathrm{nm}}\right)^2\left[\frac{l_0(2l-l_0)}{l(l-l_0)}\right]^2. \qquad (46)$$

Thus a spontaneously emitted photon shows a very small linewidth. The power radiated into space is from Eqs. (23) and (42)–(44)

$$P = \frac{\hbar\omega_{21}}{2\tau\cosh^2 t/\tau}. \qquad (47)$$

The shape of this type of photon and the dynamic occupation of the two electron levels is shown in Fig. 2. The linewidth in the Fourier spectrum is approximately given by $1/(2\tau)$, of course. It agrees with the result obtained from current theories [5] for dipole interaction

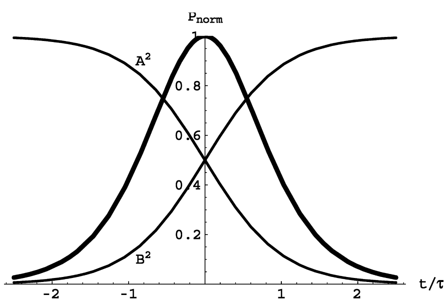


Fig. 2. Spontaneously radiated normalized energy $P_{\mathrm{norm}} = 2P\tau/(\hbar\omega_{21})$ per unit time (power, thick line) of a photon as a function of time according to Eq. (47). $A^2$ and $B^2$ are the occupation probabilities (42)–(44) of the electron on the higher and lower energy level, respectively

with light; the dipole matrix element (not to be confounded with the permeability $\mu$) is calculated to be

$$|\mu_{\mathrm{dp}}| = \frac{4ed_x}{\pi^2} \cdot \frac{l(l - l_0)}{l_0^2(2l - l_0)^2}, \tag{48}$$

under the condition that $l_0$ is an odd number, again in agreement with a result derived earlier in the present article. For even values of $l_0$ the transition probability (and $\mu_{\mathrm{dp}}$) is zero. It is of interest to estimate the maximum electric field which occurs at $t = 0$. From Eqs. (44) and (14) we find

$$\begin{aligned}\hat{E}_{\max} &= \frac{\pi^3}{6} \frac{e\hbar^3 \mu^{3/2} \varepsilon^{1/2}}{m^3 d_x^5} l\, l_0(2l - l_0)(l - l_0) \\ &\approx 5.39 \frac{\mathrm{V}}{\mathrm{m}} l\, l_0(2l - l_0)(l - l_0)\left(\frac{\mathrm{nm}}{d_x}\right)^5. \end{aligned} \tag{49}$$

The fact that the photon takes an infinitely long onset to finally take off in its essential part within $2\tau$ followed by an infinitely long trickling time, is typical for perturbation-free nonlinear systems on the brink towards instability. Any perturbation, however small, could shorten the tiny tails or prevent the take-off. Quite a similar behavior is exhibited by a classical mechanical mass experiencing a quasi-elastic restoring force proportional to $\xi(1 - \xi^2)$, with $\xi$ as normalized displacement from the stable steady state, when moving unperturbed from the unstable steady state $\xi = -1$ to $\xi = +1$: the trajectory $\xi(t)$ is given by a hyperbolic tangens function of time, as is the integral of Eq. (47), the radiated energy of the photon.

### B. Photon under dominance of an external field (stimulated emission)

Here the case of a strong constant-wave stimulation, $\hat{E} \gg \hat{E}_{\max}$ is considered with $\alpha$ from Eq. (14) being time-independent. The solutions of Eqs. (15) and (16) for this case are simply periodic and read

$$A = \cos \alpha t, \quad B = \sin \alpha t. \tag{50}$$

A power of

$$P = \hbar\omega_{21} \alpha \sin 2\alpha t \tag{51}$$

is radiated by the quantum box in the time interval $0 < t < \pi/(2\alpha)$ and absorbed in the time interval $\pi/(2\alpha) < t < \pi/\alpha$ such as to pump

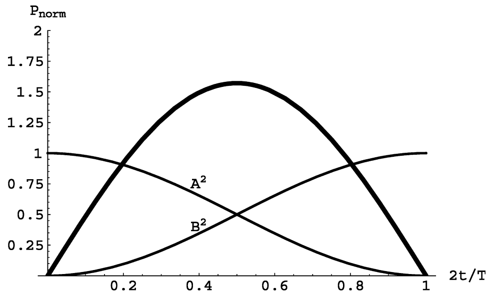


Fig. 3. Radiated normalized energy $P_{\rm norm} = PT/(2\hbar\omega_{21})$ per unit time (power, thick line) of a photon under stimulation as a function of time according to Eq. (51). $A^2$ and $B^2$ are the occupation probabilities (50) of the electron on the higher and lower energy level, respectively

the electron back to its original level. The process is repeated periodically with the period, calculated with the aid of Eq. (14),

$$T = \frac{\pi}{\alpha} = \frac{\pi^3\hbar}{4ed_x\hat{E}}\frac{l_0^2(2l-l_0)^2}{l(l-l_0)} \approx \frac{5.1\cdot 10^{-6}{\rm s}}{\frac{d_x}{\rm nm}\frac{\hat{E}}{\rm V/m}}\frac{l_0^2(2l-l_0)^2}{l(l-l_0)}. \tag{52}$$

The shape of the stimulated photon and the partition of the electron between the two levels is shown in Fig. 3. The difference to the photon emitted spontaneously, Fig. 2, is remarkable. When calculating the Fourier spectrum of a photon far from its source, the following relations [3] between radiated power of a monochromatic Hertzian dipole, $P_H$, and the far-field amplitudes in spherical coordinates $(r, \vartheta, \phi)$ have to be taken into account:

$$\hat{E}_\vartheta = \left(\frac{3}{4\pi}\cdot\left(\frac{\mu}{\varepsilon}\right)^{1/2}\cdot P_H\right)^{1/2}\cdot\frac{\sin\vartheta}{r} = \left(\frac{\mu}{\varepsilon}\right)^{1/2}\hat{H}_\phi. \tag{53}$$

Applied to the present case, the field amplitudes in the time domain show an envelope proportional to $(\sin 2\pi t/T)^{1/2}$ with $0 \le t \le T/2$. The corresponding Fourier spectrum of a single photon is shown by the solid curve in Fig. 4. A base band appears with a width of about $6/T$, and side bands show widths of about $3/T$. The author cannot share the view that the line width of a single photon emitted through stimulation agrees with that of a spontaneously emitted photon, $1/(2\tau)$. The slow natural decay is caused by the weak field produced by the emitting photon itself, given by Eqs. (14) and (44). Here the electric field is orders of magnitude larger. With an infinitely long sequence of photons the spectrum breaks up into discrete lines of zero

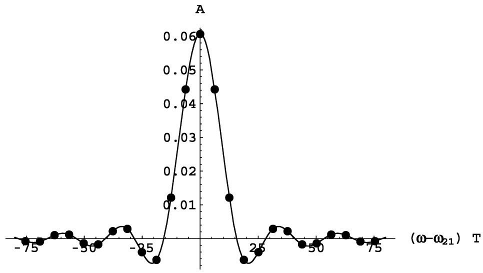


Fig. 4. Amplitude spectrum of a single photon emitted by strong stimulation (solid curve) and of an infinitely long sequence of such photons (dots), with the dots indicating location and relative strength of the spectral lines (*A* in arbitrary units)

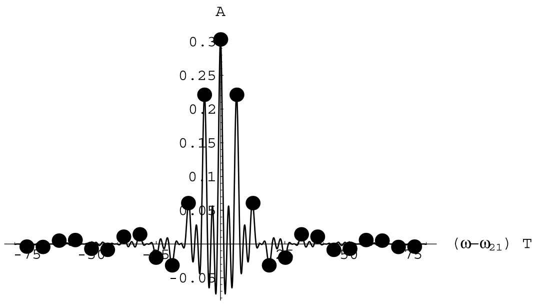


Fig. 5. Amplitude spectrum of five photons emitted in sequence (*A* in same units as in Fig. 4, except for the dots which are lifted by a factor of 5)

width located at $\omega = \omega_{21} \pm \frac{2\pi n}{T}$, with $n$ as integer, described by a Fourier series; the dots in Fig. 4 show location and relative strength of these spectral lines. The transition from the broad-band single photon to a small-band sequence of photons is demonstrated by Fig. 5, the spectrum for five photons, where all side bands show up clearly, quite well agreeing with current high-field theories [6].

The electric field acting on the electron is a collective field associated with all photons even though only one of them can be absorbed or emitted. $T/2$ is the apparent time the captured photon has to dwell on the quantum box before re-emission. Taking as an example $d_x = 10\,\text{nm}$, $l_0 = 1$, $l = 3$, and $\hat{E} = 5\hat{E}_{\text{max}}$, Eq. (52) leads to $T/2 = 1.31 \cdot 10^{-4}\,\text{s}$, a time in which a "free" photon would travel about 39.4 km.

What is the balance of power for the incident wave, thought to be a plane TEM wave? Take $n_0$ as the number of quantum boxes per unit volume, which for the sake of simplicity are assumed to be uniformely oriented in space. In the emission interval $0 < t < \pi/(2\alpha)$, the photon energy $\hbar\omega_{21}n_0$ is radiated per volume element, half of which in the direction of the incident wave, the other half is radiated back into the source impedance $(\mu/\varepsilon)^{1/2}$ and dissipated, provided that the source is far enough away. Thus the power loss $\Delta P$ of the incident wave caused per volume element is given by the loss of the energy $\hbar\omega_{21}n_0$ in the absorption interval, reduced by the gain of $\hbar\omega_{21}n_0/2$ in the emission interval, resulting in

$$\Delta P = \frac{\hbar\omega_{21}n_0}{2T} = \frac{2ed_x n_0 \omega_{21}\hat{E}}{\pi^3}\frac{l(l-l_0)}{l_0^2(2l-l_0)^2}. \tag{54}$$

Thus the apparent conductivity of the medium would be

$$\kappa = \frac{2\Delta P}{\hat{E}^2} = \frac{4n_0 ed_x\omega_{21}}{\pi^3\hat{E}}\frac{l(l-l_0)}{l_0^2(2l-l_0)^2}, \tag{55}$$

which formally leads to a power-absorption coefficient $\kappa(\mu/\varepsilon)^{1/2}$. Note, however, that the decay of the incident wave is principally a nonlinear effect and therefore non-exponential (in the validity range of Eq. (55), the electric-field amplitude can be shown to drop linearly with distance, the magnetic-field amplitude along a hyperbola, and the radiated power along a parabola). Equation (55) reflects a reduction of absorption at high densities of the electric field known in the literature as absorption saturation effect [5]. A true saturation would be caused by the effect that at increasing stimulating field, the photon may become so short in time $T/2$ that in the averaging process leading to Eq. (23), $\sin^2\omega_{21}t$ cannot simply be replaced by $1/2$. Thus $W = \hbar\omega_{21}$ is not unconditionally valid, corroborating a result obtained by Keller [2]. Calculating the total energy of the photon again from the products of Eqs. (21) and (22) under the condition that Eq. (26), $W = \int_{-\infty}^{+\infty} P dt = \hbar\omega_{21}$, has to stay valid, we reach the requirement

$$\omega_{21}T = (2N-1)\pi = \frac{\pi^5\hbar^2}{8emd_x^3\hat{E}}\frac{l_0^3(2l-l_0)^3}{l(l-l_0)}, \quad N > 1, \tag{56}$$

with $N \neq 1$ as an integer. $N = 1$ has to be excluded since it would lead to $\alpha = \omega_{21}$, a condition not allowed when reexamining the derivation of Eqs. (15) and (16) from Eqs. (9) and (10). Thus the shortest photon would consist only of a $3/4$ period of an oscillation within its envelope (with $N = 2$). It looks as if a new quantization rule with a quantum number $N$ becomes visible for very short photon-duration times.

Alternatively, if the condition were imposed that a photon has to contain full periods of oscillations (here $N = 1$ is allowed),

$$\omega_{21} T = 4\pi N, \tag{57}$$

then Eq. (26) would become invalid and had to be replaced by

$$W = \frac{\hbar\omega_{21}}{1 - (1/(4N))^2} > \hbar\omega_{21}, \tag{58}$$

which qualitatively agrees with Eq. (106) and Fig. 3 of [2] but yields lower values, which may be caused by the different models used. The author prefers to keep $W = \hbar\omega_{21}$ valid and to accept the necessary quantization rule, Eq. (56). In both cases, there is a lower limit for $T = \pi/\alpha$, and from Eq. (14), an upper limit for the electric field strength to succeed in stimulating the photon emission. With

$$\omega_{21} T_{\min} = N_{\min}\pi, \tag{59}$$

$N_{\min} = 3$ if $W = \hbar\omega_{21}$ is maintained and $N_{\min} = 4$ if full periods are required, the extreme value of the electric field amplitude for stimulating photon emission is

$$\begin{aligned} \hat{E}_{\text{extr}} &= \frac{\pi^4\hbar^2}{8emd_x^3}\frac{1}{N_{\min}}\frac{l_0^3(2l - l_0)^3}{l(l - l_0)} \\ &\approx 9.38 \cdot 10^8 \frac{\text{V}}{\text{m}}\left(\frac{\text{nm}}{d_x}\right)^3 \frac{1}{N_{\min}}\frac{l_0^3(2l - l_0)^3}{l(l - l_0)}. \end{aligned} \tag{60}$$

How could the quantization rule predicted by Eq. (56) be tested by experiment? One possibility exists through observation of absorption represented by conductivity, Eq. (55), as a function of the stimulating field. Since interaction leading to absorption occurs only at field states given by Eq. (56), corresponding absorption lines should show up. Figure 6 demonstrates the simple relation, but indicates also difficulties: Only for very high values of the stimulating field a separation between quantum states appears principally observable, but the absorption is very low. For conventional field strengths, the quantum states are so close to each other that a quasi-continuum exists. Estimates show that the extreme value of field strength, Eq. (60), can be reached with a power density of, say, $10^{12}$ W/cm$^2$, which is not out of the world, but the difficulty probably lies in the other high-field effects, such as ionization, more than two states involved in the interaction, etc., which may overlap the quantization phenomenon

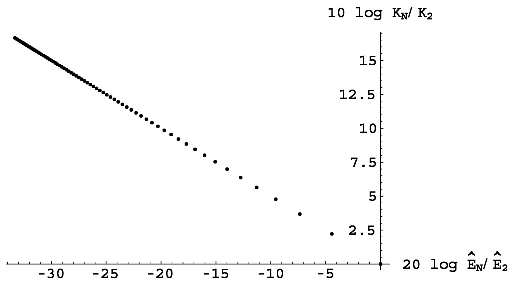


Fig. 6. Quantized states of conductivity, Eq. (55), as a function of power of the stimulating field. The stimulating field is normalized to the extreme value, i.e., the state $N = 2$, the conductivity to the corresponding value for this state

sought for. If absorption is limited to quantum states, so is stimulated emission: At very large fields it should occur only at discrete fields given by Eq. (56).

## IV. Shapes, Dispersion, and Uncertainties

The photon is shaped by its environment (here, the impedance of space around its source for the spontaneously emitted photon and the electric field of an incoming photon for the stimulation of a photon's emission). So is the electron, whose spectral components are being rearranged by dispersion (inherent to material waves), as well as reflection and evanescence caused by barriers. The high-frequency components of an electron-wave packet are able to spill over the barriers, and frequency components in the evanescent region, particularly their high-frequency part, may tunnel through the barrier. Both effects lead to a rearrangement of the spectral components such as to cause the spectrum of a passing electron to shift toward higher frequencies (energies) at reduced bandwidth, which is associated with an increased uncertainty in time. There is a reflected part consisting predominantly of the lower frequency components, so that the electron, if reflected, experiences a shift in the spectrum towards lower frequencies (energies), under observation of the uncertainty principle.

The photon emitted spontaneously had been calculated on the basis that the radiation resistance of the Hertzian dipole exists for all times. Suppose a perfect reflector is placed at a radius $r$; then for a time $2r/c$ the condition of Eq. (41) prevails, but then the first reflection is back at the source and starts terminating the power flow. Thus for $t > 2r/c$, Eq. (41) becomes invalid and, since the critical time falls into the onset of emanation, the radiation process is interrupted. Since in the following discussion time intervals are considered which are long compared to the emanation time but short to the time the reflections take to get back to the source, the photon emitted spontaneously is excluded from further discussions.

The photon is, according to the present theory, emanated with a constant angular frequency $\omega_{21}$ within an envelope, whose shape and extension, here $T/2$ for stimulated emission, determine the bandwidth of its Fourier spectrum. There is no uncertainty of energy, it has to be $\hbar\omega_{21}$ and stay at that level. As far as a rearrangement of the spectrum by dispersion, reflection, and evanescence is concerned, there is no reason why the photon should behave differently, in principle, from the electron: the spectrum can be reshuffled, here under obedience of Maxwellian rules. If a photon meets a barrier, for example produced by a periodic structure of dielectric materials (a filter for electromagnetic waves), the high-frequency components may lie in an upper pass-band and spill over the barrier formed by a stop-band, through which other components may pass by tunnelling. There is a reflected fraction of the photon, which travels in opposite direction to the passing fraction. Such a separation should be of no concern, since in the dipolar shape of a photon, the shell containing the photon's mass expands rapidly. There is no need to talk about probabilities to find the particle at a certain time at a certain place, such as in the case of the electron. The photon's shape appears as determined by the distribution of the electromagnetic field. Are $A^2$ and $B^2$ probabilistic or deterministic distribution functions? The conservation of energy, a reformulation of Eq. (23),

$$\frac{d}{dt}(\hbar\omega_2 A^2(t) + \hbar\omega_1 B^2(t)) + P(t) = 0, \tag{61}$$

does not require a probabilistic interpretation, but does not inhibit it. There is no need to interpret $A^2$ as the *probability* to observe the electron at the time $t$ on level $\hbar\omega_2$ and $B^2$ the *probability* to observe the electron on level $\hbar\omega_1$. In the authors's opinion, $A^2$ is the part of the electron which, at the time $t$, is with *certainty* on level $\hbar\omega_2$ and $B^2$ the part of the electron which is with *certainty* on level $\hbar\omega_1$. The result for stimulated emission speaks for the deterministic interpretation: The electron *has* to leave level 2 at $t = 0$ and *has* to land at level 1 at $t = T/2$ (see Fig. 3). It is forced to do so by the well-defined external field. In the case of spontaneous emission, stochastic processes are expected to cause perturbations which, as indicated in Section III-A, were disregarded in the theory but will certainly influence the radiation process.

On the other hand, the solution of the Schrödinger equations presented here is far from being rigorous: All non-resonant terms on the right-hand sides of Eqs. (2) and (3) were neglected. This casts some doubts on the efficacy of the theory to obtain far-reaching conclusions even though all results appear as physically convincing.

It should be remarked, that a calculation of the interaction of a very short photon with a periodic structure made of dielectric materials requires unconventional methods, since it is restricted to few polarizable particles; it is not possible in such a case to take polarization collectively into account by a dielectric constant.

## V. Conclusions

The electron changes its quantum state only by an odd quantum number. The photon emanated by the electron consists of constant frequency oscillations within an envelope. The photon emitted by stimulated emission is of other shape than the photon emitted spontaneously and has a much larger linewidth that increases with the square root of intensity (power). The high-frequency components of the Fourier spectrum are associated with the limited time-duration of the photon and are absolutely necessary to satisfy the energy relation $W = \hbar\omega_{21}$. Thus the bandwidth of the Fourier spectrum is not limited. Dispersion, reflection and evanescence may reshape the photon.

Stimulated emission of photons occurs only for a stimulating electric field below a certain very high level. When approaching the limit, photons become very short and cause considerations about the validity of $W = \hbar\omega_{21}$ in such an extreme case, although the formula can be maintained under certain quantization conditions for the duration of the photon oscillations.

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