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On Continuous Solutions of a Conditional Gołąb-Schinzel Equation

By

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Abstract

We determine the continuous solutions $f:(0,+\infty)\to\mathbb{R}$ of equation (3). The proof of the result is based on the well known theorem of J. Aczél, describing the continuous cancellative associative operations on a real interval.

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Let $\mathbb{R}$ denote the set of reals. In connection with a problem raised by P. Kahlig, J. Aczél and J. Schwaiger [3] have determined the continuous solutions $f:\mathbb{R}\to\mathbb{R}$ of the following conditional Gołąb-Schinzel equation

$$f(x+f(x)y)=f(x)f(y) \quad \text{for } x,y\in[0,+\infty). \tag{1}$$

A more general conditional version of the Gołąb-Schinzel equation has been studied by L. Reich [13] (see also [14]). He has found the continuous solutions $f:[0,+\infty)\to\mathbb{R}$ of the functional equation

$$f(x+f(x)y)=f(x)f(y) \quad \text{whenever } x,y,x+f(x)y\in[0,+\infty). \tag{2}$$

For more details concerning the Gołąb-Schinzel equation refer to [2]–[5], [12]–[15] and [7].

We present a different and shorter proof of the results in [3] and [13]. Moreover, we consider the following generalization of (1) and (2):

$$f(x+f(x)y)=f(x)f(y) \quad \text{whenever } x,y,x+f(x)y\in\mathbb{R}^+, \tag{3}$$

where $\mathbb{R}^+=(0,+\infty)$.

The main tool in our proof is the given below well known theorem of J. Aczél [1] (see also [11]).

Theorem 1. *Let L be a nontrivial real interval and let $\circ : L\times L\to L$ be a continuous cancellative associative operation. Then there exists a continuous bijection $h : L\to J$ such that*

$$x\circ y=h^{-1}(h(x)+h(y)) \quad \text{for } x,y\in L,$$

where J is a (*necessarily unbounded*) *real interval.*

For some further examples of applications of the Aczél Theorem in solving some functional equations of similar type see [6], [8], [9], [10].

The next theorem contains the main result of this paper.

Theorem 2. *Let $f:\mathbb{R}^+\to\mathbb{R}$ be a continuous solution of* (3). *Then either $f\equiv 0$ or there exists $c\in\mathbb{R}$ such that one of the following two conditions holds*:

1° $f(x)=cx+1$ *for* $x\in\mathbb{R}^+$;
2° $f(x)=\sup\{cx+1,0\}$ *for* $x\in\mathbb{R}^+$.

Proof: Suppose that $f(w)\neq 0$ for some $w>0$. Take a sequence $\{x_n\}\subset\mathbb{R}^+$ with $x_n\to 0$. Then

$$f(w)=f(\lim w+f(w)x_n)=\lim f(w+f(w)x_n)=\lim f(w)f(x_n)$$

and consequently

$$1=f(w)^{-1}\lim f(w)f(x_n)=\lim f(x_n).$$

Whence

$$\lim_{y\to 0^+} f(y)=1. \tag{4}$$

Let $B:=f^{-1}(\{0\})$. Assume that $B\neq\emptyset$ and put $a=\inf B$. From (4) we infer $a>0$. Let $I=(0,a)$ and $A=\{x+f(x)a : x>0\}$. Then $f(a)=0$,

$$f(x)>0 \quad \text{for } x\in I, \tag{5}$$

$$A \text{ is connected.} \tag{6}$$

Since $a = a + f(a)a \in A$ and

$$f(x + f(x)a) = f(x)f(a) = 0 \quad \text{whenever } x + f(x)a > 0, \tag{7}$$

from (5) and (6) we derive

$$a \in A \subset [a, +\infty). \tag{8}$$

If $x + f(x)a = a$ for every $x > 0$, we obtain 1° with $c = -a^{-1}$. So it remains to consider the cases: (i) $B \neq \emptyset$ and $A \setminus \{a\} \neq \emptyset$; (ii) $B = \emptyset$.

Suppose that (i) holds and $f(y) \neq 0$ for some $y > a$. Then, by (6), (7) and (8), there is $b \in (a, y)$ such that

$$[a, b] \subset A \subset B. \tag{9}$$

Put $D = \{x + f(x)y : x \in I\}$. Note that $x + f(x)y \to a$ if $x \to a^-$ and, on account of (4), $x + f(x)y \to y$ if $x \to 0^+$; whence $b \in (a, y) \subset D$. This brings a contradiction, because, by (5), $0 \notin f(I)f(y) = f(D)$ and, by (9), $f(b) = 0$. Consequently $B = [a, +\infty)$.

Thus we have proved that, in either case, $L := \mathbb{R}^+ \setminus B$ is an interval. So, by (4), $f(L) \subset \mathbb{R}^+$ and consequently $f(x + f(x)y) = f(x)f(y) \neq 0$ for $x, y \in L$. Hence $x + f(x)y \in L$ for $x, y \in L$, which means that we may define a binary operation $\circ : L \times L \to L$ by:

$$x \circ y = x + f(x)y.$$

Take $x, y, z \in L$. If $x \circ y = x \circ z$, then clearly $y = z$. Next, if

$$y + f(y)x = y \circ x = z \circ x = z + f(z)x, \tag{10}$$

then $0 \neq f(y)f(x) = f(y \circ x) = f(z \circ x) = f(z)f(x)$ and consequently $f(y) = f(z)$; whence, by (10), $y = z$. Finally,

$$\begin{aligned}(x \circ y) \circ z &= x + f(x)y + f(x + f(x)y)z \\ &= x + f(x)(y + f(y)z) = x \circ (y \circ z).\end{aligned}$$

So we have shown that the operation is cancellative and associative. Since it is continuous, by Theorem 1, it is commutative, which means that

$$x + f(x)y = y + f(y)x \quad \text{for } x, y \in L. \tag{11}$$

Fix $y \in L$ and put $c = (f(y) - 1)y^{-1}$. Then (11) implies

$$f(x) = cx + 1 \quad \text{for } x \in L.$$

This yields the statement.

Remark. Having proved that L is an interval, instead of applying the Aczél Theorem, we may complete the proof of Theorem 2 in the same way as in [3], using [3, Lemmas 1 and 2] (then we need only to consider the case $f(x_0) > 0$ in the proof of Lemma 1).

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