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On a Linear Diophantine Equation, II

By

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Let for vectors $\boldsymbol{a} = [a_1, \dots, a_{\mathrm{K}}] \in \mathbb{Z}^{\mathrm{K}}$, $\boldsymbol{x} = [x_1, \dots, x_{\mathrm{K}}] \in \mathbb{Z}^{\mathrm{K}}$, $r(\boldsymbol{x}) = \Pi_{l=1}^{\mathrm{K}} \max\{1, |x_i|\}$, $\boldsymbol{a} \cdot \boldsymbol{x} = a_1 x_1 + \dots + a_{\mathrm{K}} x_{\mathrm{K}}$.

M. Drmota [2] has proved the following theorem.

For every $k \geq 1$ and $\boldsymbol{a} \in (\mathbb{Z} \setminus \{0\})^{k+1}$ there exists a non-zero integral solution $\boldsymbol{x}$ of the equation $\boldsymbol{a} \cdot \boldsymbol{x} = 0$ with

$$r(\boldsymbol{x}) \leq k r(\boldsymbol{a})^{1/k}.$$

A. Schinzel and the author [1] have proved that for every $k \geq 1$ there exist a positive constant $C(k)$ and vectors $\boldsymbol{a}$ with arbitrarily large $r(\boldsymbol{a})$ such that for every $\boldsymbol{x} \in \mathbb{Z}^{k+1} \setminus \{\mathbf{0}\}$ the equation $\boldsymbol{a} \cdot \boldsymbol{x} = 0$ implies

$$r(\boldsymbol{x}) \geq C(k) r(\boldsymbol{a})^{1/(k)}, \quad C(k) > 0.$$

It is natural to ask for the best value of the coefficient $C(k)$.

In this paper we shall consider the case $k = 2$ and we shall answer this question by proving two theorems.

Theorem 1. *For every vector* $\boldsymbol{a} \in \mathbb{Z}^3$, $1 \leq a_1 < a_2 < a_3$, $\boldsymbol{a} \neq [1, 2, 4]$ *there exists a non-zero vector* $\boldsymbol{x} \in \mathbb{Z}^3$ *such that*

$$\boldsymbol{a} \cdot \boldsymbol{x} = 0 \quad \textit{and} \quad r(\boldsymbol{x}) < \frac{1}{\sqrt[4]{5}} \sqrt{a_1 a_2 a_3}. \tag{1}$$

Theorem 2. *For every $\varepsilon > 0$ there exist infinitely many vectors $\boldsymbol{a} \in \mathbb{Z}^3$ such that for every $\boldsymbol{x} \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$ the equation $\boldsymbol{a} \cdot \boldsymbol{x} = 0$ implies*

$$r(\boldsymbol{x}) > \left(\frac{1}{\sqrt[4]{5}} - \varepsilon\right)\sqrt{a_1 a_2 a_3}. \tag{2}$$

Proof of Theorem 1: Let $\alpha = \frac{a_3}{a_1 a_2}$, $\Pi = a_1 a_2 a_3$, $D = D(a_1, a_2, a_3) = \frac{1}{\sqrt[4]{5}}\sqrt{\Pi}$.

First of all let us consider two particular cases.

The first, $\boldsymbol{a} = [1, a_2, a_3]$.
If $\alpha > \sqrt{5}$, then $\boldsymbol{x} = [a_2, -1, 0]$,
if $1 < \alpha \le 1.92$, then $\boldsymbol{x} = [a_3 - a_2, 1, -1]$,
if $1.58 \le \alpha \le 2.53$, then $\boldsymbol{x} = [a_3 - 2a_2, 2, -1]$.

The second, $\boldsymbol{a} = [2, a_2, a_3]$.

If a_2 or a_3 is even, e.g. $2 \mid a_2$, then for the vector $\boldsymbol{a}' = [1,\ a_2/2,\ a_3] \ne [1, 2, 4]$ there exists a vector $\boldsymbol{x}' = [x_1', x_2', x_3']$ satisfying (1). And thus $\boldsymbol{x} = [x_1', x_2', 2x_3']$ satisfies (1) for the vector $\boldsymbol{a} = [2,\ a_2,\ a_3]$.

Let thus a_2, a_3 be odd.

If $\alpha > \sqrt{5}$, then $\boldsymbol{x} = [a_2, -2, 0]$,
if $1/2 < \alpha < \sqrt{5}$, then $\boldsymbol{x} = [(a_3 - a_2)/2, 1, -1]$.

Now let us observe that the validity of Thm. 1 for $\boldsymbol{a} = [a_1, a_2, a_3]$ implies its validity for the vectors $[da_1, da_2, a_3]$, $[da_1, a_2, da_3]$, $[a_1, da_2, da_3]$, where $d \in \boldsymbol{N}$. Likewise, for $d > 1$ and the vectors $[d, 2d, 4]$, $[d, 2, 4d]$, $[1, 2d, 4d]$ Theorem 1 is fulfilled.

As a consequence, assume further on that

$$a_1 \ge 3, \tag{3}$$

and

$$(a_i, a_j) = 1, \quad i \ne j. \tag{4}$$

I. $\alpha > \sqrt{5}$.

Then $\boldsymbol{x} = [a_2,\ -a_1,\ 0]$ satisfies (1).

II. $\frac{1}{7} < \alpha < \sqrt{5}$.

By Minkowski's theorem on linear forms there exists a non-zero vector $\boldsymbol{x} \in \mathbb{Z}^3$ such that

$$\begin{cases} |a_1 x_1 + a_2 x_2 + a_3 x_3| < 1, \\ \qquad |a_2 x_2 + \frac{1}{2} a_3 x_3| \le \frac{1}{2} a_1 a_2, \\ \qquad\qquad |x_3| < 2. \end{cases}$$

Hence choosing $x_3 \geq 0$,

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 = 0, & (5)\\ \left|a_2x_2 + \frac{1}{2}a_3x_3\right| \leq \frac{1}{2}a_1a_2, & (6)\\ x_3 = 0 \text{ or } 1. \end{cases}$$

If we had $x_1x_2x_3 = 0$, then in virtue of (4) and (5) we should obtain a contradiction with (6).

Therefore, we have $|x_1| \geq 1$, $|x_2| \geq 1$, $x_3 = 1$.

(i) $x_1 \geq 1$, $x_2 \leq -1$. Then from (5) and (6) $x_1 = (-a_2x_2 - a_3x_3)/a_1 \geq 1$ and

$$-\frac{1}{2}a_1a_2 - \frac{1}{2}a_3x_3 \leq a_2x_2 \leq \min\left\{\frac{1}{2}a_1a_2 - \frac{1}{2}a_3x_3, -a_3x_3 - a_1\right\},$$

hence $\alpha < 1$ and

$$\begin{aligned} r(\boldsymbol{x}) &= \frac{(-a_2x_2 - a_3x_3)(-a_2x_2)}{a_1a_2} \\ &\leq \frac{\left(\frac{1}{2}a_1a_2 - \frac{1}{2}a_3x_3\right)\left(\frac{1}{2}a_1a_2 + \frac{1}{2}a_3x_3\right)}{a_1a_2} = \frac{(a_1a_2)^2 - a_3^2}{4a_1a_2} \\ &= \frac{a_1a_2(1-\alpha^2)}{4} = \frac{1-\alpha^2}{4\sqrt{\alpha}}\sqrt{\Pi} < D. \end{aligned}$$

(ii) $x_1 \leq -1$, $x_2 \geq 1$. Then from (6)

$$\max\left\{-\frac{1}{2}a_1a_2 - \frac{1}{2}a_3x_3, a_2\right\} \leq a_2x_2 \leq \frac{1}{2}a_1a_2 - \frac{1}{2}a_3x_3,$$

hence $\alpha < 1$ and just as above we obtain $r(\boldsymbol{x}) < D$.

(iii) $x_1 \leq -1$, $x_2 \leq -1$. Then

$$\begin{aligned} r(\boldsymbol{x}) &= \frac{(a_2x_2 + a_3x_3)(-a_2x_2)}{a_1a_2} \leq \frac{a_3^2}{4a_1a_2} \\ &= \frac{\alpha^{\frac{3}{2}}}{4}\sqrt{\Pi} < D, \quad \text{for } \alpha < \alpha_0 = \sqrt[6]{51.2}. \end{aligned}$$

Further on, in this case, it is sufficient to assume that $\alpha_0 < \alpha < \sqrt{5}$. Let

$$q_1 = \frac{-x_1}{a_2}, \quad q_2 = \frac{-x_2}{a_1}.$$

Then $q_1, q_2 > 0$, $q_1 + q_2 = \alpha$ and $\boldsymbol{x} = [x_1, x_2, 1] = [-q_1a_2, -q_2a_1, 1] = [-(\alpha - q_2)a_2, -q_2a_1, 1]$.

If $\boldsymbol{x}$ does not satisfy (1), then for $\Delta = \alpha^2 - 4\sqrt{\alpha}/\sqrt[4]{5}$

$$q_2 \in \left(\frac{\alpha - \sqrt{\Delta}}{2}, \frac{\alpha + \sqrt{\Delta}}{2}\right) \subset \left(\frac{\alpha - 1}{2}, \frac{\alpha + 1}{2}\right),$$

because $\Delta < 1$.

a) $(\alpha-1)/2 < q_2 \leq \alpha/2$. Let $\boldsymbol{y} = [-(\alpha-1-q_2)a_2, -(q_2+1)a_1, 1]$. Obviously, $\boldsymbol{y} = [x_1+a_2, x_2-a_1, 1] \in \mathbb{Z}^3$ and $\boldsymbol{a} \cdot \boldsymbol{y} = 0$. We shall show that $\boldsymbol{y}$ satisfies (1).

a1) $q_2 > \alpha-1$. Hence, $\alpha < 2$ and

$$\begin{aligned} r(\boldsymbol{y}) &= \frac{(-\alpha + 1 + q_2)(q_2 + 1)}{\sqrt{\alpha}}\sqrt{\Pi} \leq \frac{(1 - \frac{\alpha}{2})(1 + \frac{\alpha}{2})}{\sqrt{\alpha}}\sqrt{\Pi} \\ &= \frac{4 - \alpha^2}{4\sqrt{\alpha}}\sqrt{\Pi} < D, \quad \text{for } \alpha \geq 3/2. \end{aligned}$$

a2) $q_2 < \alpha-1$. Hence, $\alpha > 1$ and

$$r(\boldsymbol{y}) = \frac{(\alpha - 1 - q_2)(q_2 + 1)}{\sqrt{\alpha}}\sqrt{\Pi} < \frac{\alpha^2 - 1}{4\sqrt{\alpha}}\sqrt{\Pi} < D.$$

a3) $q_2 = \alpha - 1$. Then $\boldsymbol{y} = [0, -\alpha a_1, 1] = [0, -a_3/a_2, 1]$, hence $a_3/a_2 \in \mathbb{Z}$, a contradiction with (4).

b) $\alpha/2 \leq q_2 < (\alpha + 1)/2$. Let $\boldsymbol{y} = [-(\alpha + 1 - q_2)a_2, -(q_2 - 1), 1]$. Obviously, $\boldsymbol{y} = [x_1 - a_2, x_2 + a_1, 1] \in \mathbb{Z}^3$ and $\boldsymbol{a} \cdot \boldsymbol{y} = 0$. We shall show that $\boldsymbol{y}$ satisfies (1).

b1) $q_2 > 1$. Then

$$r(\boldsymbol{y}) = \frac{(\alpha + 1 - q_2)(q_2 - 1)}{\sqrt{\alpha}}\sqrt{\Pi} \leq \frac{\frac{\alpha+1}{2} \cdot \frac{\alpha-1}{2}}{\sqrt{\alpha}}\sqrt{\Pi} = \frac{\alpha^2 - 1}{4\sqrt{\alpha}}\sqrt{\Pi} < D.$$

b2) $q_2 < 1$. Hence $\alpha < 2$ and

$$\begin{aligned} r(\boldsymbol{y}) &= \frac{(\alpha + 1 - q_2)(1 - q_2)}{\sqrt{\alpha}}\sqrt{\Pi} \leq \frac{(1 + \frac{\alpha}{2})(1 - \frac{\alpha}{2})}{\sqrt{\alpha}}\sqrt{\Pi} \\ &= \frac{4 - \alpha^2}{4\sqrt{\alpha}}\sqrt{\Pi} < D, \end{aligned}$$

for $\alpha \geq 3/2$.

b3) $q_2 = 1$. Then $\boldsymbol{y} = [-\alpha a_2, 0, 1] = [-a_3/a_1, 0, 1]$, hence $a_3/a_1 \in \mathbb{Z}$, a contradiction with (3) and (4).

III. $\alpha < 1/7$.

Let $t \in \mathbb{R}$, $t \geq 2$. By Minkowski's theorem on linear forms there exists a non-zero vector $\boldsymbol{x} \in \mathbb{Z}^3$ such that $x_3 \geq 0$ and

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 & = 0, \\ \left|a_2x_2 + \frac{1}{2}a_3x_3\right| & \leq \frac{1}{t}a_1a_2, \\ x_3 & < t. \end{cases} \tag{7}$$

Just as in case II, we obtain that $x_1x_2x_3 \neq 0$,

(i) $x_1 \geq 1$, $x_2 \leq -1$. Then from (7)

$$-\frac{1}{t}a_1a_2 - \frac{1}{2}a_3x_3 \leq a_2x_2 \leq \min\left\{\frac{1}{t}a_1a_2 - \frac{1}{2}a_3x_3, -a_3x_3 - a_1\right\},$$

hence $\alpha x_3 < 2/t$ and

$$\begin{aligned} r(\boldsymbol{x}) &= \frac{(-a_2x_2 - a_3x_3)(-a_2x_2)a_3x_3}{\Pi} \\ &\leq \frac{\left(\frac{1}{t}a_1a_2 - \frac{1}{2}a_3x_3\right)\left(\frac{1}{t}a_1a_2 + \frac{1}{2}a_3x_3\right)a_3x_3}{\Pi} \\ &= \frac{\left[\left(\frac{2}{t}\right)^2 - (\alpha x_3)^2\right]\alpha x_3}{4\alpha^{3/2}}\sqrt{\Pi} \\ &\leq \frac{\left[\left(\frac{2}{t}\right)^2 - \frac{4}{3t^2}\right]\frac{2}{\sqrt{3}t}}{4\alpha^{3/2}}\sqrt{\Pi} = \frac{4}{3\sqrt{3}\left(\sqrt{\alpha}t\right)^3}\sqrt{\Pi}. \end{aligned}$$

(ii) $x_1 \leq -1$, $x_2 \geq 1$. Then from (7)

$$\max\left\{-\frac{1}{t}a_1a_2 - \frac{1}{2}a_3x_3, a_2\right\} \leq a_2x_2 \leq \frac{1}{t}a_1a_2 - \frac{1}{2}a_3x_3,$$

hence $\alpha x_3 < 2/t$ and

$$\begin{aligned} r(\boldsymbol{x}) &= \frac{(a_2x_2 + a_3x_3)a_2x_2a_3x_3}{\Pi} \\ &\leq \frac{\left(\frac{1}{t}a_1a_2 + \frac{1}{2}a_3x_3\right)\left(\frac{1}{t}a_1a_2 - \frac{1}{2}a_3x_3\right)a_3x_3}{\Pi} \\ &= \frac{\left[\left(\frac{2}{t}\right)^2 - (\alpha x_3)^2\right]\alpha x_3}{4\alpha^{3/2}}\sqrt{\Pi} \leq \frac{4}{3\sqrt{3}\left(\sqrt{\alpha}t\right)^3}\sqrt{\Pi}. \end{aligned}$$

(iii) $x_1 \le -1$, $x_2 \le -1$. Then

$$r(\boldsymbol{x}) = \frac{(a_2x_2 + a_3x_3)(-a_2x_2)a_3x_3}{\Pi} \le \frac{(a_3x_3)^3}{4\Pi}$$
$$= \frac{a_3^3x_3^3}{4\Pi^{3/2}}\sqrt{\Pi} = \frac{\alpha^{3/2}\cdot x_3^3}{4}\sqrt{\Pi} < \frac{(\sqrt{\alpha}t)^3}{4}\sqrt{\Pi}.$$

Putting $t = \sqrt[6]{3}/\sqrt{\alpha}$ we obtain $t > 2$ and

$$r(\boldsymbol{x}) < D,$$

in every case (i), (ii), (iii).

The proof of Theorem 1 is complete.

The proof of Theorem 2 will be based on the Lemma below. Let $F_0 = 0, F_1 = 1, F_2 = 1, \dots, F_n$ be the Fibonacci sequence and let $\theta = \frac{1+\sqrt{5}}{2}$.

Lemma. *Let $[x, q] \in \mathbb{Z}^2$ and let M_n be the minimal non-zero value of the function*

$$P_n(x, q) = |F_{2n}x - F_{2n+1}q||F_{n-1}x - F_nq||x|$$
$$= F_{2n+1}F_n\left|\frac{F_{2n}}{F_{2n+1}} - \frac{q}{x}\right|\left|\frac{F_{n-1}}{F_n} - \frac{q}{x}\right||x|^3,$$

where $n \ge 3$ is odd. Then

$$M_n = P_n(F_{n+2}, F_{n+1}) = F_{n-1}F_{n+2} = F_nF_{n+1} - 1.$$

Proof: For $n = 3$,

$$P_3(x, q) = |8x - 13q||x - 2q||x|,$$

hence $M_3 = P_3(1,\ 1) = P_3(5,\ 3) = 5 = F_5F_2$.

For $n \ge 5$ we may assume that $x > q > 0$. Let us observe that

$$0 = \frac{F_0}{F_1} < \frac{F_2}{F_3} < \frac{F_4}{F_5} < \cdots < \frac{1}{\theta} < \cdots < \frac{F_5}{F_6} < \frac{F_3}{F_4} < \frac{F_1}{F_2} = 1.$$

Let k be a positive odd integer.

If

$$\frac{F_{k-3}}{F_{k-2}} < \frac{q}{x} \le \frac{F_{k-1}}{F_k}, \quad 3 \le k \le n-2,$$

then

$$\frac{F_{k-3}F_k}{F_{k-2}F_k} < \frac{q}{x} \le \frac{F_{k-2}F_{k-1}}{F_{k-2}F_k} = \frac{F_{k-3}F_k + 1}{F_{k-2}F_k},$$

hence $x \geq F_k$ and

$$\begin{aligned}
P_n(x,q) &\geq F_{2n+1}F_n\left|\frac{F_{2n}}{F_{2n+1}}-\frac{F_{k-1}}{F_k}\right|\left|\frac{F_{n-1}}{F_n}-\frac{F_{k-1}}{F_k}\right|F_k^3\\
&= |F_{2n}F_k - F_{2n+1}F_{k-1}||F_{n-1}F_k - F_nF_{k-1}|F_k\\
&= F_{2n-k+1}F_{n-k}F_k \geq F_{n+3}F_2F_{n-2} = F_{n+3}F_{n-2} > F_{n+2}F_{n-1}.
\end{aligned}$$

If

$$\frac{F_{n-3}}{F_{n-2}} < \frac{q}{x} < \frac{F_{n-1}}{F_n},$$

then $P_n(x,q) > |F_{2n}F_n - F_{2n+1}F_{n-1}|F_n = F_{n+1}F_n$.

If

$$\frac{F_{n-1}}{F_n} < \frac{q}{x} < \frac{F_{n+1}}{F_{n+2}},$$

then $P_n(x,q) > |F_{2n}F_{n+2} - F_{2n+1}F_{n+1}|F_{n+2} = F_{n-1}F_{n+2}$.

If

$$\frac{q}{x} = \frac{F_{n+1}}{F_{n+2}},$$

then $P_n(x,q) \geq |F_{2n}F_{n+2} - F_{2n+1}F_{n+1}|\ |F_{n-1}F_{n+2} - F_nF_{n+1}|F_{n+2} = F_{n-1}F_{n+2}$.

If

$$\frac{F_{k-1}}{F_k} < \frac{q}{x} \leq \frac{F_{k+1}}{F_{k+2}}, \quad n+2 \leq k \leq 2n-3,$$

then $P_n(x,q) > |F_{2n}F_{k+2} - F_{2n+1}F_{k+1}|\,|F_{n-1}F_k - F_nF_{k-1}|\,F_{k+2}\frac{F_{k+2}}{F_k} > F_{2n-k-1}F_{k-n} \cdot 2F_{k+2} \geq F_{n-3}F_2 \cdot 2F_{n+4} = 2F_{n-3}F_{n+4} = 2(F_{n-1}F_{n+2} - 5) = F_{n-1}F_{n+2} + (F_{n-1}F_{n+2} - 10) > F_{n-1}F_{n+2}$.

If

$$\frac{F_{k-1}}{F_k} < \frac{q}{x} \leq \frac{F_{k+1}}{F_{k+2}}, \quad k \geq 2n-1, \quad \frac{q}{x} \neq \frac{F_{2n}}{F_{2n+1}},$$

then $P_n(x,q) > F_{2n+1} > F_{n-1}F_{n+2}$.

Let furthermore $\frac{q}{x} > \frac{1}{\theta}$. Now it is sufficient to distinguish just yet three cases. If

$$\frac{F_{k+2}}{F_{k+3}} \leq \frac{q}{x} < \frac{F_k}{F_{k+1}}, \quad 1 \leq k \leq n-4,$$

then $P_n(x,q) \geq |F_{2n}F_{k+3} - F_{2n+1}F_{k+2}|\ |F_{n-1}F_{k+3} - F_nF_{k+2}|F_{k+3} = F_{2n-k-2}F_{n-k-3}F_{k+3} \geq F_{n+2}F_{n-1}$.

If

$$\frac{F_{k+2}}{F_{k+3}} \leq \frac{q}{x} < \frac{F_k}{F_{k+1}}, \quad n-2 \leq k \leq 2n-3,$$

then $P_n(x,q) \geq |F_{2n}F_{k+3} - F_{2n+1}F_{k+2}|\ |F_{n-1}F_{k+3} - F_nF_{k+2}|F_{k+3} = F_{2n-k-2}F_{k+3-n}F_{k+3} \geq F_nF_{n+1}$.

If

$$\frac{F_{k+2}}{F_{k+3}} \leq \frac{q}{x} < \frac{F_k}{F_{k+1}}, \quad k \geq 2n-1,$$

then $P_n(x,q) > F_{k+3} \geq F_{2n+2} > F_{n-1}F_{n+2}$.

The Lemma has been proved.

Proof of Theorem 2: For every $n \in \mathbb{N}$, the following formulas

$$(F_n + F_{n+2})F_n = F_{2n} + (-1)^{n+1},$$
$$(F_n + F_{n+2})F_{n+1} = F_{2n+2} = F_{2n} + F_{2n+1}, \qquad (8)$$

and

$$F_nF_{2n} = F_{n-1}F_{2n+1} + (-1)^{n+1}F_{n+1}, \qquad (9)$$

are true. Therefore the equality

$$F_nx_1 + F_{n+1}x_2 + F_{2n+1}x_3 = 0, \qquad (10)$$

takes, for n odd the form

$$x_1 + F_{2n}x_2 - F_{2n+1}q = 0,$$

where $q = -x_1 - x_2 - (F_n + F_{n+2})x_3$, i.e.

$$x_1 = -F_{2n}x_2 + F_{2n+1}q,$$

hence putting to (10) and using (9) we have

$$x_3 = F_{n-1}x_2 - F_nq.$$

As a consequence, for $\boldsymbol{a} = [F_n, F_{n+1}, F_{2n+1}]$, n odd, $\boldsymbol{x} = [x_1, x_2, x_3]$,

$$|x_1|\ |x_2|\ |x_3| = |F_{2n}x_2 - F_{2n+1}q|\ |F_{n-1}x_2 - F_nq|\ |x_2|.$$

Now, we apply the Lemma with $x = x_2$ and the proof of Theorem 2 is complete.

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