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Recurrences for Some Sequences of Binomial Sums

By

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Abstract

We prove that the sequences

$$a(n,i,l,m,z) = \sum_{k\in\mathbb{Z}} \binom{n}{\left\lfloor \frac{n+ik+l}{m} \right\rfloor} z^k \in \mathbb{Q}[z,z^{-1}]$$

satisfy a linear recurrence of order i with constant coefficients and show how these coefficients can be computed.

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Key words: Recurrence, Lucas polynomial, Fibonacci polynomial, binomial coefficient, binomial sum, difference operator.

0. Introduction

Our starting point are the following remarkable identities for the Fibonacci numbers F_n (cf. [1] and [5]):

$$F_n = \sum_{k\in\mathbb{Z}} (-1)^k \binom{n}{\left\lfloor \frac{n+5k+2}{2} \right\rfloor} \tag{0.1}$$

and

$$F_{n+1} = \sum_{k \in \mathbb{Z}} (-1)^k \binom{n}{\left\lfloor \frac{n+5k}{2} \right\rfloor} \tag{0.2}$$

The purpose of this paper is to put these identities into a general context in order to give an "explanation" of these formulas. We prove that the sequences $a(n,i,l,m,z) = \sum_{k \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+ik+l}{m} \rfloor} z^k$ satisfy a linear recurrence of order i with constant coefficients and show how these coefficients can be computed. Some special cases have previously been obtained by G. E. Andrews [1] with other methods.

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1. Sums of the Form $\sum_{k \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+ik+l}{2} \rfloor} z^k$

We study first the polynomials

$$a(n,i,l,z) = a(n,i,l,2,z) = \sum_{k \in \mathbb{Z}} \binom{n}{\left\lfloor \frac{n+ik+l}{2} \right\rfloor} z^k \in \mathbb{Q}[z, z^{-1}], \tag{1.1}$$

where $i \geq 1$ and l are integers.

It is easy to see that

$$a(n,2,l,z) = (1+z)^n z^{-\lfloor \frac{n+l}{2} \rfloor} \tag{1.2}$$

There are some obvious relations between some of these polynomials. E.g. we have

$$a(n,i,l,z) = a(n,2i,l,z^2) + za(n,2i,l+i,z^2) \tag{1.3}$$

For

$$\begin{aligned} a(n,i,l,z) &= \sum \binom{n}{\left\lfloor \frac{n+l+2ki}{2} \right\rfloor} z^{2k} + \\ &+ \sum \binom{n}{\left\lfloor \frac{n+l+(2k+1)i}{2} \right\rfloor} z^{2k+1} = \\ &= a(n,2i,l,z^2) + za(n,2i,l+i,z^2). \end{aligned}$$

A trivial relation is

$$a(n,i,l-i,z)=za(n,i,l,z). \tag{1.4}$$

Therefore we only need to consider $l\in\{0,1,\ldots,i-1\}$.

It follows that

$$\begin{aligned}a(n,1,l,z)&=a(n,2,l,z^2)+za(n,2,l+1,z^2)=\\&=(1+z^2)^n\left(z^{-2\lfloor\frac{n+l}{2}\rfloor}+z^{1-2\lfloor\frac{n+l+1}{2}\rfloor}\right)=\\&=(1+z)(1+z^2)^n z^{-n-l}.\end{aligned}$$

Thus

$$a(n,1,l,z)=(1+z)(1+z^2)^n z^{-n-l}. \tag{1.5}$$

In other cases we cannot hope to find explicit formulas. But it turns out that they satisfy simple recurrence relations.

We shall prove the following

Theorem 1.1. *The sequence of the polynomials $(a(n,i,l,z))_{n\geq 0}$ satisfies the homogeneous recurrence*

$$\sum_{j\leq\frac{i}{2}}(-1)^j\binom{i-j}{j}\frac{i}{i-j}a(n+i-2j,i,l,z)=\left(z+\frac{1}{z}\right)a(n,i,l,z) \tag{1.6}$$

with constant coefficients.

Before proving this theorem let us make some remarks.

For $i=1$ we get the recurrence $a(n+1,1,l,z)=\left(z+\frac{1}{z}\right)\cdot a(n,1,l,z)$. Since $a(0,1,l,z)=\binom{0}{\lfloor\frac{-l}{2}\rfloor}z^{-l}+\binom{0}{\lfloor\frac{-l+1}{2}\rfloor}z^{-l+1}=z^{-l}(1+z)$ we get (1.5).

For $i=2$ the recurrence reduces to

$$a(n+2,2,l,z)=\frac{(z+1)^2}{z}a(n,2,l,z).$$

This is a recurrence of order 2. Therefore we need 2 initial values. These are $a(0,2,l,z)=z^{-\lfloor\frac{l}{2}\rfloor}$, $a(1,2,l,z)=(1+z)z^{-\lfloor\frac{l+1}{2}\rfloor}$. Hence (1.2) holds.

Everyone familiar with Fibonacci numbers recognizes the coefficients on the left hand side of (1.6). These occur in the Lucas polynomials $L_n(x,s)$ which are defined by

$$L_n(x,s)=xL_{n-1}(x,s)+sL_{n-2}(x,s),\ n\geq 2, \tag{1.7}$$

with initial values $L_0(x,s)=2$, $L_1(x,s)=x$.

They are explicitly given by $L_n(x,s)=\sum_{j<n}\binom{n-j}{j}\frac{n}{n-j}x^{n-2j}s^j$. From (1.7) we see by induction that

$$L_n(x,1-x)=1+(x-1)^n \tag{1.8}$$

holds.

Consider now the vector space of all functions f on the nonnegative integers and define the translation operator E and the finite difference operator $\Delta=E-I$ as usual by $Ef(n)=f(n+1)$ and $\Delta f(n)=f(n+1)-f(n)$. Then (1.8) implies

$$L_i(E,-\Delta)f(n)=f(n)+\Delta^i f(n). \tag{1.9}$$

For $f(n)=\binom{n}{r}$ this reduces to

$$\begin{aligned}\binom{n}{r}+\binom{n}{r-i}&=L_i(E,-\Delta)\binom{n}{r}=\\&=\sum_{j<i}(-1)^j\binom{i-j}{j}\frac{i}{i-j}\binom{n+i-2j}{r-j}.\end{aligned} \tag{1.10}$$

This holds for all values $r\in\mathbb{Z}$. As a special case we get

$$\begin{aligned}&\binom{n}{\left\lfloor\frac{n+i(k+1)+l}{2}\right\rfloor}+\binom{n}{\left\lfloor\frac{n+i(k-1)+l}{2}\right\rfloor}=\\&\quad=\sum_{j<i}(-1)^j\binom{i-j}{j}\frac{i}{i-j}\binom{n+i-2j}{\left\lfloor\frac{n+i(k+1)-2j+l}{2}\right\rfloor}=\\&\quad=\sum_{j<i}(-1)^j\binom{i-j}{j}\frac{i}{i-j}E^{i-2j}\binom{n}{\left\lfloor\frac{n+ik+l}{2}\right\rfloor}.\end{aligned}$$

If we multiply both sides by z^k and sum over all $k\in\mathbb{Z}$ we get (1.6). Thus our theorem is proved.

In order to give some impression of the operators

$$A_i(E)=\sum_{j<i}(-1)^j\binom{i-j}{j}\frac{i}{i-j}E^{i-2j}=L_i(E,-1)$$

we explicitly state the first ones:

$$A_1(E) = E$$
$$A_2(E) = E^2 - 2$$
$$A_3(E) = E^3 - 3E$$
$$A_4(E) = E^4 - 4E^2 + 2$$
$$A_5(E) = E^5 - 5E^3 + 5E$$
$$A_6(E) = E^6 - 6E^4 + 9E^2 - 2$$

For $i = 5$ and $z = -1$ the theorem tells us that sequence $b(n,l) = a(n,5,l,-1)$ satisfies the recurrence $(E^5 - 5E^3 + 5E + 2)\, b(n,l) = 0$. From $E^5 - 5E^3 + 5E + 2 = (E+2)(E^2 - E - 1)^2$ it is clear that the Fibonacci numbers satisfy the same recurrence.

Computing the initial values immediately gives $b(n,0) = b(n,1) = F_{n+1}$ and $b(n,2) = F_n$, thus proving (0.1) and (0.2).

Remark:
If we only want to prove that the sequence of polynomials $(a(n,i,l,z))_{n\geq 0}$ satisfies a homogeneous linear recurrence with constant coefficients of order i, it suffices to show the following

Lemma 1.1. *For each integer $n \geq 1$ there exist integers such that $1 + x^n = \sum a_{n,j}(x+1)^{n-2j}x^j$ holds.*

For this implies that $1 + \Delta^n = \sum a_{n,j}E^{n-2j}\Delta^j$ and the argument continues as above.

In order to prove the lemma observe that for each j, $0 \leq j \leq \frac{n}{2}$, the polynomial $(x+1)^{n-2j}x^j$ is symmetric about $\frac{n}{2}$. If we eliminate the coefficients of x^j, $1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$, in $(x+1)^n$ we get therefore $1 + x^n$.

It turns out that for $z = 1$ and $z = -1$ there are simpler recurrences. We consider first the case $z = -1$.

First we notice that because of (1.4) we have $a(n,i,l+i,-1) = -a(n,i,l,-1)$. Therefore we need only consider $l \in \{0,1,\ldots,i-1\}$. For small values of i we can find explicit values for these sums. Thus we have $a(n,1,l,-1) \equiv 0$ for all $l \in \mathbb{Z}$ and $a(n,2,0,-1) = a(n,2,1,-1) = [n=0]$, where the symbol $[P]$ equals 1 if the assertion P holds and it is 0 if P does not hold.

Further we have $a(n,\,3,\,0,\,-1) = a(n,\,3,\,1,\,-1) \equiv 1$ and $a(n,\,3,\,2,\,-1) \equiv 0$, $a(n,\,4,\,0,\,-1) = a(n,\,4,\,1,\,-1) = 2^{\lfloor \frac{n}{2} \rfloor}$, $a(n,\,4,\,2,\,-1) = -a(n,\,4,\,3,\,-1) = 0$ for even n, $a(n,\,4,\,2,\,-1) = -a(n,\,4,\,3,\,-1) = 2^{\lfloor \frac{n-1}{2} \rfloor}$ for n odd, and finally $a(n,\,5,\,0,\,-1) = a(n,\,5,\,1,\,-1) = F_{n+1}$, $a(n,\,5,\,2,\,-1) = -a(n,\,5,\,4,\,-1) = F_n$, and $a(n,\,5,\,3,\,-1) \equiv 0$.

It turns out that our results differ for even and odd values of i.

2. Sums of the Form $\sum_{k\in\mathbb{Z}}(-1)^k\binom{n}{\lfloor\frac{n+2ik+l}{2}\rfloor}$

We first prove

Theorem 2.1. *The sequences $(a(n,2i,l,-1))$ satisfy the homogeneous linear recurrence*

$$\sum_{j<i}(-1)^j\binom{i-j}{j}\frac{i}{i-j}E^{i-2j}a(n,2i,l,-1)=0 \tag{2.1}$$

Remark. If we denote the operator on the lefthand side by

$$B_{2i}(E)=\sum_{j<i}(-1)^j\binom{i-j}{j}\frac{i}{i-j}E^{i-2j}=L_i(E,-1),\text{ then}$$
$$B_{2i}(E)=A_i(E).$$

The sequence $(a(n,2i,l,-1))$ satisfies therefore both the recurrences

$$A_i(E)a(n,2i,l,-1)=0\text{ and }(A_{2i}(E)+2I)a(n,2i,l,-1)=0.$$

In order to understand this situation note that

$$L_{2i}(x,-1)+2=(L_i(x,-1))^2.$$

This follows immediately from the well-known formula (cf. e.g. [3])

$$L_n(x,s)=\left(\frac{x+\sqrt{x^2+4s}}{2}\right)^n+\left(\frac{x-\sqrt{x^2+4s}}{2}\right)^n. \tag{2.2}$$

In order to prove (2.1) we observe that

$$(I+\Delta^i)\binom{n}{\lfloor\frac{r+2ik+l}{2}\rfloor}=\binom{n}{\lfloor\frac{r+2ik+l}{2}\rfloor}+$$
$$+\binom{n}{\lfloor\frac{r+2i(k-1)+l}{2}\rfloor}.$$

This implies $(I+\Delta^i)a(n,2i,l,-1)=0$, because it turns out to be a telescoping sum.

Now we know that $1 + \Delta^i = \sum a_{i,j} E^{i-2j} \Delta^j$ for some coefficients $a_{i,j}$. If we apply $E^{i-2j}\Delta^j$ to $\binom{n}{\lfloor \frac{r+l+2ik}{2} \rfloor}$ we get

$$\binom{n+i-2j}{\left\lfloor \frac{r+i-2j+l-i+2ik}{2} \right\rfloor}.$$

Hence we get $\sum a_{i,j} a(n, 2i, l-i, -1) = 0$. Since this holds for each l, our theorem is proved.

3. Sums of the Form $\sum_{k \in \mathbb{Z}} (-1)^k \binom{n}{\lfloor \frac{n+(2i-1)k+l}{2} \rfloor}$

In order to study $a(n, 2i-1, l, -1)$ we observe that

$$a(n, 2i-1, i+l, -1) = -a(n, 2i-1, i-l, -1). \qquad (3.1)$$

Since $n = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor$ we have

$$n - \left\lfloor \frac{n+(2i-1)k-l+i}{2} \right\rfloor = \left\lfloor \frac{n-(2i-1)(k+1)-i+l}{2} \right\rfloor.$$

This means that the term with index k in $a(n, 2i-1, i-l, -1)$ is the opposite of the term with index $-(k+1)$ in $a(n, 2i-1, i+l, -1)$.

As special cases we get

$$a(n, 2i-1, i, -1) = 0 \qquad (3.2)$$

and

$$a(n, 2i-1, 1, -1) = -a(n, 2i-1, 2i, -1) = a(n, 2i-1, 0, -1) \qquad (3.3)$$

Therefore we get

$$\sum_{j=1}^{2i-1} a(n, 2i-1, j, -1) = 0. \qquad (3.4)$$

But we also have

$$\sum_{j=1}^{i-1} a(n, 2i-1, 2j, -1) = 0, \qquad (3.5)$$

because

$$a(n, 2i-1, 2i-2j, -1) = a(n, 2i-1, i+(i-2j), -1) =$$
$$= -a(n, 2i-1, i-(i-2j), -1) = -a(n, 2i-1, 2j, -1).$$

Thus also

$$\sum_{j=1}^{i} a(n,2i-1,2j-1,-1) = -\sum_{j=1}^{i} a(n,2i-1,2j-1+2i-1,-1) = 0$$

which implies $\sum_{j=1}^{2i-1} a(n,2i-1,2j,-1) = 0$.

As a consequence we get that the sum over $2i-1$ consecutive values of the form $a(n,2i-1,l-2j,-1)$ vanishes and therefore we have

$$\sum_{j=-i+1}^{i-1} a(n,2i-1,l-2j,-1) = 0 \tag{3.6}$$

for each $l \in \mathbb{Z}$.

This is equivalent with

$$\sum_{j=0}^{i-1} a(n,2i-1,l-2j,-1) - \sum_{j=1}^{i-1} a(n,2i-1,l+1-2j,-1) = 0 \tag{3.7}$$

Each term in $a(n,2i-1,l,-1)$ has the form $f(n,r,k,l) = (-1)^k \binom{n}{\lfloor \frac{r+l+(2i-1)k}{2} \rfloor}$ with $r = n$ and $a(n,2i-1,l,-1)$ is a finite sum of expressions $f(n,n,k,l) - f(n,n,k+1,l)$.

Given r and l we may choose one of two consecutive k's such that $r + l + (2i-1)k$ is even.

Then we have

$$\begin{aligned} f(n,r,k,l-2j) - f(n,r,k+1,l-2j) &= \\ = \Delta^j (f(n,r,k,l) - f(n,r,k+1,l)) \end{aligned} \tag{3.8}$$

If we set

$$b(n,i,l,r) = \sum_{k\in\mathbb{Z}} (-1)^k \binom{n}{\lfloor \frac{r+ik+l}{2} \rfloor},$$

then we have

$$\begin{aligned} 0 &= \sum_{j=-i+1}^{i-1} a(n,2i-1,l-2j,-1) = \\ &= (1+\Delta+\cdots+\Delta^{i-1}) b(n,2i-1,l,r)_{r=n} - \\ &\quad - \Delta(1+\Delta+\cdots+\Delta^{i-2}) b(n,2i-1,l+1,r)_{r=n} \end{aligned}$$

If we define the Filbonacci polynomials $F_n(x,s)$ by the recurrence $F_n(x,s) = xF_{n-1}(x,s) + sF_{n-2}(x,s)$ with the initial values $F_0(x,s) = 0$ and $F_1(x,s) = 1$ then we have

$$F_n(x,s) = \sum_{j \le \frac{n-1}{2}} \binom{n-1-j}{j} x^{n-1-j} s^j.$$

It is immediately verified that

$$F_n(x, 1-x) = 1 + (x-1) + \cdots + (x-1)^{n-1}$$

holds.

Therefore (3.7) may be formulated as

$$\begin{aligned}(F_i(E,-\Delta)b(n,2i-1,l,r) - \Delta F_{i-1}(E,-\Delta)\cdot \\ \cdot\, b(n,2i-1,l+1,r))_{r=n} = 0\end{aligned} \tag{3.9}$$

Now we have

$$\begin{aligned}E^{-2j}\Delta^j \left(\binom{n}{\left\lfloor \frac{r+l+(2i-1)k}{2} \right\rfloor} \right)_{r=n} &= \\ = \binom{n-2j}{\left\lfloor \frac{n-2j+l+(2i-1)k}{2} \right\rfloor} &= \\ = E^{-2j} \binom{n}{\left\lfloor \frac{n+l+(2i-1)k}{2} \right\rfloor}\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}E^{-2j-1}\Delta^{j+1} \left(\binom{n}{\left\lfloor \frac{r+l+1+(2i-1)k}{2} \right\rfloor} \right)_{r=n} &= \\ = \binom{n-2j-1}{\left\lfloor \frac{n-2j-1+l+(2i-1)k}{2} \right\rfloor} &= \\ = E^{-2j-1} \binom{n}{\left\lfloor \frac{n+l+(2i-1)k}{2} \right\rfloor}\end{aligned} \tag{3.11}$$

Therefore (3.9) is equivalent with

$$(F_i(E,-1) - F_{i-1}(E,-1))a(n,2i-1,l,-1)) = 0.$$

This implies

Theorem 3.1. *The sequences* $(a(n,2i-1,l,-1))$ *satisfy the homogeneous linear recurrence*

$$(F_i(E,-1)-F_{i-1}(E,-1))a(n,2i-1,l,-1))=0 \tag{3.12}$$

The operators $C_i(E)=F_i(E,-1)-F_{i-1}(E,-1)$ are for small values i explicitly given by

$$\begin{aligned}
C_1(E)&=1\\
C_2(E)&=E-1\\
C_3(E)&=E^2-E-1\\
C_4(E)&=E^3-E^2-2E+1\\
C_5(E)&=E^4-E^3-3E^2+2E-1
\end{aligned}$$

As an example consider $i=3$. We get

$$a(n+2,5,l,-1)-a(n+1,5,l,-1)-a(n,5,l,-1)=0.$$

Thus $a(n,5,l,-1)$ satisfies the recurrence of the Fibonacci numbers. By choosing the appropriate initial values we get again formulas (0.1) and (0.2).

To understand the relation between Theorem 1 and Theorem 3 we note that

$$(F_n(x,-1)-F_{n-1}(x,-1))^2(x+2)=L_{2n-1}(x,-1)+2$$

holds. This follows from the well-known formula (cf. [3])

$$F_n(x,s)=\frac{1}{\sqrt{x^2+4s}}\left(\left(\frac{x+\sqrt{x^2+4s}}{2}\right)^n-\left(\frac{x-\sqrt{x^2+4s}}{2}\right)^n\right) \tag{3.13}$$

in connection with (2.2).

4. Sums of the Form $\sum_{k\in\mathbb{Z}}\binom{n}{\lfloor\frac{n+2ik+l}{2}\rfloor}$

Simpler recurrences than in the general case are also possible for $z=1$.

We must again distinguish between even and odd i.

Let us first consider the sums $a(n,2i,l,1)$. Here we have

$$(\Delta^i-1)\sum\binom{n}{r+ik}=0.$$

We know already that $F_n(x, 1-x) = 1 + (x-1) + \cdots + (x-1)^{n-1}$ holds and therefore we may write

$$\Delta^i - 1 = (E-2)F_i(E, -\Delta).$$

For $n + l \equiv 1 (\bmod 2)$ we have

$$E^{-2j} \binom{n}{\left\lfloor \frac{n+2ik+l}{2} \right\rfloor} = E^{-2j} \Delta^j \binom{n}{r}_{r=\left\lfloor \frac{n+2ik+l}{2} \right\rfloor}$$

and

$$E^{-2j-1} \binom{n}{\left\lfloor \frac{n+2ik+l}{2} \right\rfloor} = E^{-2j-1} \Delta^j \binom{n}{r}_{r=\left\lfloor \frac{n+2ik+l}{2} \right\rfloor}.$$

If $n + l \equiv 0 (\bmod 2)$ we have

$$\binom{n}{\left\lfloor \frac{n+2ik+l}{2} \right\rfloor} = \binom{n}{\left\lfloor \frac{n+2ik-l+1}{2} \right\rfloor}.$$

Thus in each case we get

$$(E-2)F_i(E, -1)a(n, 2i, l, 1) = 0.$$

This gives

Theorem 4.1. *The sequences* $(a(n, 2i, l, 1))$ *satisfy the homogeneous linear recurrence*

$$(E-2)F_i(E, -1)a(n, 2i, l, 1) = 0 \tag{4.1}$$

These operators are for small values of i given by

$$\begin{aligned} D_1(E) &= E - 2 \\ D_2(E) &= E^2 - 2E \\ D_3(E) &= E^3 - 2E^2 - E - 2 \\ D_4(E) &= E^4 - 2E^3 - 2E^2 + 4E \end{aligned}$$

Again this is connected to the formula

$$L_{2i}(x, -1) - 2 = (x^2 - 4)(F_i(x, -1))^2$$

5. Sums of the Form $\sum_{k\in\mathbb{Z}} \binom{n}{\lfloor \frac{n+(2i-1)k+l}{2} \rfloor}$

Now remains the case $a(n, 2i-1, l, 1)$.
Our aim is the following theorem.

Theorem 5.1. *The sequences* $(a(n, 2i-1, l, 1))$ *satisfy the homogeneous linear recurrence*

$$(L_i(E, -1) - L_{i-1}(E, -1))a(n, 2i-1, l, 1) = 0.$$

In order to prove this observe that

$$a(n, 2i-1, l, 1) = a(n, 2i-1, l+2i-1, 1)$$

and therefore

$$a(n, 2i-1, l, 1) + a(n, 2i-1, l-2i, 1) = \\ = (a, n, 2i-1, l-1, 1) + a(n, 2i-1, l+1-2i, 1) \quad (5.1)$$

The same reasoning as in (3.8) gives

$$(1 + \Delta^i)a(n, 2i-1, l, 1) = \Delta(1 + \Delta^{i-1})a(n, 2i-1, l+1, 1). \quad (5.2)$$

From (3.10) and (3.11) and (1.9) we see that (5.2) means

$$(L_i(E, -1) - L_{i-1}(E, -1))a(n, 2i-1, l, 1) = 0.$$

Thus our theorem is proved.

We note again the corresponding formula for the Lucas polynomials

$$(x-2)(L_{2n-1}(x, -1) - 2) = (L_n(x, -1) - L_{n-1}(x, -1))^2.$$

6. Sums of the Form $\sum_{k\in\mathbb{Z}} \binom{n}{\lfloor \frac{n+ik+l}{3} \rfloor} z^k$

We first prove the

Lemma 6.1. *For each integer* $n \geq 1$ *there exist integers such that*

$$1 + (-1)^{n-1}x^n = \sum a_{n,j}(x+1)^{n-3j}x^j + \sum b_{n,j}(x+1)^{2n-3j}x^j \quad (6.1)$$

holds.

Let us make some comments. We are looking for a term of the form $1 + a(n)x^n$ which occurs on the right hand side, i.e. which is a linear combination of the polynomials $(x+1)^{n-3j}x^j$, $0 \leq j \leq \frac{n}{3}$ and

$(x+1)^{2n-3j}x^j$, $\frac{n}{2} \le j \le \frac{2n}{3}$. The only polynomial of this form which contains 1 is $(x+1)^n$. For odd n this is also the only one containing x^n. Thus $a(n)$ must be 1 in this case.

If $n = 2m$ there is another polynomial of this form which contains x^n, namely $(x+1)^m x^m = (x+1)^{4m-3m} x^m$. In order to obtain a linear combination of the form $1 + a(n)x^n$ we must find b such that the coefficient of x^{n-1} in $(x+1)^{2m} + b((x+1)^m x^m$ equals 0. Thus $b = -2$ and the coefficient of x^n must therefore be -1.

Proof of Lemma 6.1:
To prove this define polynomials

$$v_n(x,s) = xv_{n-1}(x,s) - sv_{n-3}(x,s) \tag{6.2}$$

with initial values

$$v_0(x,s) = 3, v_1(x,s) = x, v_2(x,s) = x^2 \tag{6.3}$$

Then $v_n(x,s) = \sum_{3j\le n} a_{n,j} x^{n-3j} s^j$ for some coefficients $a_{n,j}$.

This is easily proved by induction.

It is not difficult to determine these polynomials explicitly: For $n > 0$ we have

$$v_n(x,s) = \sum_{3j\le n} (-1)^j \binom{n-2j}{j} \frac{n}{n-2j} x^{n-3j} s^j. \tag{6.4}$$

For the recurrence is easily verified and the initial values coincide for $n = 1, 2, 3$.

Consider now

$$v_n(x+1,x) = \sum_{3j\le n} a_{n,j} (x+1)^{n-3j} x^j.$$

These polynomials satisfy the recurrence

$$v_n(x+1,x) = (x+1)v_{n-1}(x+1,x) - xv_{n-3}(x+1,x).$$

The characteristic polynomial of this recurrence is $\lambda^3 - (x+1)\lambda^2 + x$ with $\lambda = 1$ as one root. Dividing this polynomial by $\lambda - 1$ we get $\lambda^2 - x\lambda - x$.

Now we know that the Lucas polynomials

$$L_n(x,x) = \sum \binom{n-j}{j} \frac{n}{n-j} x^{n-j} = \sum_i \binom{i}{n-i} \frac{n}{i} x^i$$

satisfy the recurrence $L_n(x,x) = xL_{n-1}(x,x) + xL_{n-2}(x,x)$. Therefore they also satisfy $L_n(x,x) = (x+1)L_{n-1}(x,x) - xL_{n-3}(x,x)$.

The initial values are $L_0(x,x) = 2$, $L_1(x,x) = x$.

Since the constant polynomials also satisfy this recurrence we see that $v_n(x+1,x) = L_n(x,x) + 1$.

Next we define polynomials

$$w_n(x,s) = xsw_{n-2}(x,s) + s^2 w_{n-3}(x,s) \tag{6.5}$$

with initial values

$$w_0(x,s) = 3, \quad w_1(x,s) = 0, \quad w_2(x,s) = 2xs \tag{6.6}$$

Then $w_n(x,s) = \sum_{3j \le 2n} b_{n,j} x^{2n-3j} s^j$ for some coefficients $b_{n,j}$. This may be proved by induction.

For $n > 0$ these polynomials are explicitly given by

$$w_n(x,s) = \sum_{3j \le 2n} \binom{n-j}{2n-3j} \frac{n}{n-j} x^{2n-3j} s^j. \tag{6.7}$$

This is again easily verified.

Consider now

$$w_n(x+1,x) = \sum_{3j \le 2n} b_{n,j} (x+1)^{2n-3j} x^j.$$

These polynomials satisfy the recurrence

$$w_n(x+1,x) = x(x+1)w_{n-2}(x+1,x) + x^2 w_{n-3}(x+1,x).$$

The characteristic polynomial of this recurrence is $\lambda^3 - x(x+1)\lambda - x^2$ with $\lambda = -x$ as one root. Dividing this polynomial by $\lambda + x$ we get again $\lambda^2 - x\lambda - x$.

Thus $w_n(x+1,x) = L_n(x,x) + a(-x)^n$.

Comparing the initial values we see that $a = 1$.

Therefore we get

$$v_n(x+1,x) - w_n(x+1,x) = 1 - (-x)^n.$$

Thus our lemma is proved.

Therefore we also have

$$v_i(E,\Delta) - w_i(E,\Delta) = 1 + (-1)^{i-1}\Delta^i \tag{6.8}$$

for all i.

If we apply this to $\binom{n}{r}$ we get

$$\binom{n}{r}+(-1)^{i-1}\binom{n}{r-i}=\sum a_{i,j}\binom{n+i-3j}{r-j}+\sum b_{i,j}\binom{n+2i-3j}{r-j}$$

Now let $r=\lfloor\frac{n+i(k+1)}{3}\rfloor$.
Then we have

$$\begin{aligned}\binom{n}{\lfloor\frac{n+i(k+1)}{3}\rfloor}+(-1)^{i-1}\binom{n}{\lfloor\frac{n+i(k-2)}{3}\rfloor}&=\\ =\sum a_{i,j}\binom{n+i-3j}{\lfloor\frac{n-3j+i(k+1)}{3}\rfloor}&+\\ +\sum b_{i,j}\binom{n+2i-3j}{\lfloor\frac{n-3j+i(k+1)}{3}\rfloor}&\end{aligned}$$

Multiplying this identity by z^k and summing over all k we obtain

Theorem 6.1. *The sequence*

$$a(n,i,l,3,z):=\sum_{k\in\mathbb{Z}}\binom{n}{\lfloor\frac{n+ik+l}{3}\rfloor}z^k$$

satisfies the recursion

$$(v_i(E,1)+zw_i(E,1))b(n,i,l,3,z)=\left(\frac{1}{z}+z^2\right)a(n,i,l,3,z) \quad (6.9)$$

7. Sums of the Form $\sum_{k\in\mathbb{Z}}\binom{n}{\lfloor\frac{n+ik+l}{m}\rfloor}z^k$

Now we want to prove a general result on sums of the form

$$a(n,i,l,m,z)=\sum_{k\in\mathbb{Z}}\binom{n}{\lfloor\frac{n+ik+l}{m}\rfloor}z^k.$$

We need the following

Lemma 7.1. *For each pair of integers $n \geq 1$, $m \geq 1$ there exist uniquely determined integers $a_{n,m,j}$ such that*

$$(1+x)^n + \sum_{k=1}^{m-1} \sum_{j=\lceil \frac{(k-1)n}{m-1} \rceil}^{\frac{kn}{m}} a_{n,m,j}(x+1)^{kn-mj}x^j = 1$$

or equivalently

$$(1+x)^n + \sum_{j=1}^{n} a_{n,m,j}(x+1)^{(-mj)(\bmod n)}x^j = 1, \tag{7.1}$$

where $x(\bmod n)$ denotes the least nonnegative representative of the residue class modulo n and $a_{n,m,j} = 0$ if $(-mj)(\bmod n) + j > n$.

There is a useful reformulation which has been inspired by a remark of G. Kowol (private communication): If (7.1) is true and we replace $x+1$ by ζ_n, a primitive root of unity of order n, then we get $\sum_{j=1}^{n} a_{n,m,j}\zeta_n^{-mj}(\zeta_n - 1)^j = 0$. Thus $\zeta_n^{-m}(\zeta_n - 1)$ is a root of the polynomial $\sum_{j=1}^{n} a_{n,m,j}x^j$. The most obvious polynomial having $\zeta_n^{-m}(\zeta_n - 1)$ as a root is $\sum_{j=1}^{n} b_{n,m,j}x^j = \prod_{k=1}^{n}(x - \zeta_n^{-mk}(\zeta_n^k - 1))$. If $b_{n,m,j} = 0$ for all j with $(-mj)(\bmod n) + j > n$ then the polynomials

$$\sum_{j=1}^{n} b_{n,m,j}(x+1)^{(-mj)(\bmod n)}x^j$$

and $(1+x)^n - 1$ both have degree n and the roots 0 and $\lambda_k = \zeta_n^k - 1$, $1 \leq k \leq n-1$. Therefore they are proportional. Now we have

$$b_{n,m,1} = (-1)^{n-1}\prod_{k=1}^{n-1}\zeta_n^{-mk}(\zeta_n^k - 1) = \zeta_n^{-m\binom{n}{2}}\prod_{k=1}^{n-1}(1-\zeta_n^k) =$$

$$= (-1)^{m(n-1)}\prod_{k=1}^{n-1}(1-\zeta_n^k)$$

From $\prod_{k=1}^{n-1}(x - \zeta_n^k) = \frac{x^n - 1}{x-1}$ we get $\prod_{k=1}^{n-1}(1-\zeta_n^k) = n$ and thus we have

$$b_{n,m,1} = (-1)^{m(n-1)}n.$$

Thus we get

$$(1+x)^n - (-1)^{m(n-1)}\sum_{1 \leq j \leq n} b_{n,m,j}(x+1)^{(-mj)(\bmod n)}x^j = 1 \tag{7.2}$$

and Lemma 7.1 is proved.

Therefore our assertion is reduced to

Lemma 7.2. *Let $m, n \in \mathbb{Z}$, $n > 0$ and let ζ_n be a primitive root of unity of order n. If we set*

$$\sum_{j=1}^{n} b_{n,m,j} x^j = \prod_{k=1}^{n} (x - \zeta_n^{-mk}(\zeta_n^k - 1)), \tag{7.3}$$

then $b_{n,m,j} = 0$ for all j satisfying $\left\{\frac{-mj}{n}\right\} + \frac{j}{n} > 1$ where $\{x\}$ denotes $\{x\} = x - \lfloor x \rfloor$.

The following proof is due to Prof. Andrzej Schinzel [4].

Let s_j be the j'th elementary symmetric function of

$$\zeta_n^{-mk}(\zeta_n^k - 1), \quad 1 \le k \le n,$$

and p_j the j'th powersum of those numbers.

Lemma 7.3. *If $\left\{\frac{mj}{n}\right\} > \frac{j}{n}$, then $p_j = 0, 1 \le j < n$.*

Proof: We have

$$p_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \sum_{k=1}^{n} \zeta_n^{k(i-mj)} \tag{7.4}$$

since

$$p_j = \sum_{k=1}^{n} \zeta_n^{-mkj} (\zeta_n^k - 1)^j = \sum_{k=1}^{n} \zeta_n^{-mkj} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \zeta_n^{ki} =$$
$$= \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \sum_{k=1}^{n} \zeta_n^{k(i-mj)}.$$

The inner sum is different from 0 only if $i - mj \equiv 0 (\mathrm{mod}\, n)$.

But since $i \le j$ this implies $\left\{\frac{mj}{n}\right\} \le \frac{j}{n}$, contrary to the assumption.

Proof of Lemma 7.2:

We shall prove by induction on $j \le n$ that

$$\text{either} \left\{\frac{mj}{n}\right\} \le \frac{j}{n} \text{ or } s_j = 0.$$

This is true for $j = 0$.

Assume now that it is true for all $i < j$ and that $\left\{\frac{mj}{n}\right\} > \frac{j}{n}$.

Then by Lemma 7.3 $p_j = 0$.

Newton's formula gives

$$\sum_{i=0}^{j-1}(-1)^i p_{j-i}s_i + (-1)^j j s_j = 0. \tag{7.5}$$

Since $\left\{\frac{mj}{n}\right\} > \frac{j}{n}$ we have for each $i < j$ either $\left\{\frac{mi}{n}\right\} > \frac{i}{n}$ or $\left\{\frac{m(j-i)}{n}\right\} > \frac{j-i}{n}$.

This implies by the inductive assumption either $s_i = 0$ or by the Lemma 7.3 $p_{j-i} = 0$.

Thus $\sum_{i=0}^{j-1}(-1)^i p_{j-i}s_i = 0$ and by Newton's formula $s_j = 0$.

The inductive proof is complete.

Now if $\left\{\frac{-mj}{n}\right\} + \frac{j}{n} > 1$ then $\left\{\frac{m(n-j)}{n}\right\} > \frac{n-j}{n}$.

Hence $s_{n-j} = 0$ and $b_{n,m,j} = (-1)^{n-j}s_{n-j} = 0$.

Remark:

G. Kowol pointed out to me that for $\gcd(m,n) = 1$ formula (7.1) may be written in the following form

$$(1+x)^n + \sum_{j=1}^{n-1} a_{n,m,j}(x+1)^k x^{-m^{-1}k(\operatorname{mod} n)} = 1 + (-1)^{m(n-1)}x^n,$$

if we put $k = -mj(\operatorname{mod} n)$.

Changing x into $-1-x$ we get

$$(-x)^n + \sum_{j=1}^{n-1} a_{n,m,j}(-1)^{k-m^{-1}k(\operatorname{mod} n)}x^k(1+x)^{-m^{-1}k(\operatorname{mod} n)} =$$

$$= 1 + (-1)^{m(n-1)+n}(1+x)^n$$

This gives the corresponding formula for $m^{-1}(\operatorname{mod} n)$.

More precisely we have in this case

$$a_{n,m^{-1}(\operatorname{mod} n),k} = (-1)^{j+(-mj)(\operatorname{mod} n)}a_{n,m,j},$$

with $k = -mj(\operatorname{mod} n)$.

For example we have $(a_{7,2,j})_{j=1}^{6} = (-7, 14, -7, 0, 0, 0)$. Now $2^{-1}(\operatorname{mod} 7) = 4$ and therefore we get $(a_{7,4,j})_{j=1}^{6} = (-7, 0, -14, 0, -7, 0)$.

With the same notation as in Lemma 7.2 we can now prove

Lemma 7.4. *For all $n, i, r \in \mathbb{Z}$ we have the identity*

$$\binom{n+i}{r} - (-1)^{m(i-1)}\sum_{j=1}^{i} b_{i,m,j}\binom{n+(-mj)(\operatorname{mod} i)}{r-j} = \binom{n}{r}$$

or

$$\binom{n+i}{r} - (-1)^{m(i-1)} \sum_{j=1}^{i-1} b_{i,m,j} \binom{n+(-mj)(\operatorname{mod} i)}{r-j} =$$

$$= \binom{n}{r} + (-1)^{m(i-1)} \binom{n}{r-i}$$

Proof: This may be derived by multiplying (7.2) by a power of $1+x$ and comparing coefficients or by considering the operator identity

$$E^i - (-1)^{m(i-1)} \sum_{1 \le j \le i} b_{i,m,j} E^{(-mj)(\operatorname{mod} i)} \Delta^j = 1,$$

which follows from (7.2) by the homomorphism $x \to \Delta$ and applying it to $\binom{n}{r}$.

Now let $r = \lfloor \frac{n+i(k+1)}{m} \rfloor$.

Then we have

$$\binom{n}{\lfloor \frac{n+i(k+1)}{m} \rfloor} + (-1)^{m(i-1)} \binom{n}{\lfloor \frac{n+i(k-m+1)}{m} \rfloor} =$$

$$= \binom{n+i}{\lfloor \frac{n+i(k+1)}{m} \rfloor} - (-1)^{m(i-1)} \cdot$$

$$\cdot \sum_{j=1}^{i-1} b_{i,m,j} \binom{n+(-mj)(\operatorname{mod} i)}{\lfloor \frac{n+i(k+1)}{m} \rfloor - j}$$

Multiplying this identity by z^k and summing over all k we see that the sequence $a(n,i,l,m,z) = \sum_{k \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+ik+l}{m} \rfloor} z^k$ satisfies the recursion

$$\sum_{k \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+l+i(k+1)}{m} \rfloor} z^k + (-1)^{m(i-1)} \cdot$$

$$\cdot \sum_{k \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+l+i(k-m+1)}{m} \rfloor} z^k =$$

$$= \sum_{k\in\mathbb{Z}} \binom{n+i}{\lfloor \frac{n+l+i(k+1)}{m} \rfloor} z^k -$$

$$- (-1)^{m(i-1)} \sum_{j=1}^{i-1} b_{i,m,j} \sum_{k\in\mathbb{Z}} \binom{n+(-mj)(\operatorname{mod} i)}{\lfloor \frac{n+l+i(k+1)-jm}{m} \rfloor} z^k$$

This gives

Theorem 7.1. *Define* $h(j)$ *by* $h(j)i = mj + (-mj)\operatorname{mod} i - i$. *The sequence*

$$a(n,i,l,m,z) = \sum_{k\in\mathbb{Z}} \binom{n}{\lfloor \frac{n+ik+l}{m} \rfloor} z^k$$

satisfies the recursion

$$(E^i - (-1)^{m(i-1)} \sum_{j=1}^{i-1} b_{i,m,j} z^{h(j)} E^{(-mj)\operatorname{mod} i} - \frac{1}{z} -$$

$$- (-1)^{m(i-1)} z^{m-1}) a(n,i,l,m,z) = 0.$$

As an example consider $a(n,8,0,6,z) = \sum_{k\in\mathbb{Z}} \binom{n}{\lfloor \frac{n+8k}{6} \rfloor} z^k$.

In this case we have

$$1 + x^8 = (1+x)^8 - 8x(1+x)^2 - 12x^2(1+x)^4 + 2x^4 - 8x^5(1+x)^2$$

and therefore we get the recurrence

$$\left(E^8 - 8E^2 - 12zE^4 + 2z^2 - 8z^3E^2 - \frac{1}{z} - z^5\right) a(n,8,0,6,z) = 0.$$

8. Concluding Remarks

We have seen that there are integers $a_{n,m,j}$ such that

$$(1+x)^n + \sum_{j=1}^{n} a_{n,m,j} (x+1)^{(-mj)(\operatorname{mod} n)} x^j = 1$$

holds.

For $m = 1, 2, 3$ we know the explicit values of the coefficients $a_{n,m,j}$. Professor Schinzel has informed me that for n odd and $m = \frac{n+1}{2}$ explicit values may be obtained from formula (8) in [2]. There it is shown that $(x+1)^{2n+1}-1=\sum_{j=0}^{n} s_j x^{2j+1}(x+1)^{n-j}$ where $\sum_{j=0}^{n} s_j z^j=\prod_{j=1}^{n}\left(z+2\left(1-\cos\frac{2\pi j}{2n+1}\right)\right)$ and $s_j=\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}$.

Here we have $m=n+1$ since

$$n-j\equiv-(n+1)(2j+1)\bmod(2n+1).$$

This is a special case of our results because we have

$$\sum_{j=0}^{n} s_j z^{2j}=\prod_{k=1}^{2n}(z+\zeta^{-(n+1)k}(1-\zeta^k))=$$

$$=\prod_{j=1}^{2n}(z+\zeta^{nk}-\zeta^{-nk})=\prod_{j=1}^{n}(z+\zeta^{nk}-\zeta^{-nk})\prod_{j=1}^{n}(z-(\zeta^{nk}-\zeta^{-nk}))=$$

$$=\prod_{k=1}^{n}(z^2-(\zeta^{nk}-\zeta^{-nk})^2)=\prod_{k=1}^{n}\left(z^2-2\cos\frac{2\pi nk}{2n+1}+2\right).$$

From $-nk\equiv(n+1)k(\bmod 2n+1)$ we see that this also equals $\prod_{k=1}^{n}\left(z^2-2\cos\frac{2\pi k}{2n+1}+2\right)$.

In the general case we have only some special explicit results.

First we show that Newton's formula gives a recursion for the coefficients $b_{n,m,j}$.

Lemma 8.1. *Let* $d_{n,m,j}=(-1)^{j-(mj)(\bmod n)}n\binom{j}{(mj)(\bmod n)}$. *For* $j<n$ *we have*

$$b_{n,m,n-j}=-\frac{1}{j}\sum_{i=0}^{j-1} b_{n,m,n-i}d_{n,m,j-i} \tag{8.1}$$

with initial value $b_{n,m,n}=1$.

Proof: (7.4) gives $p_j=d_{n,m,j}$, $j<n$.

Finally we want to show how the formulas (6.4) and (6.7) for $m=3$ can be generalized to arbitrary m to give explicit values for some coefficients.

To do this it is convenient to define $0(\bmod n)=n$ and $a_{n,m,0}=1$.

Lemma 8.2. $a_{n,m,j}=(-1)^j\binom{n-(m-1)j}{j}\frac{n}{n-(m-1)j}$ *for* $0\le j\le\frac{n}{m}$.

Proof:
It suffices to show that in

$$\sum_{j=0}^{\frac{n}{m}}(-1)^j\binom{n-(m-1)j}{j}\frac{n}{n-(m-1)j}(x+1)^{n-mj}x^j$$

all coefficients of x^j up to $j=\left\lfloor\frac{n}{m}\right\rfloor$ vanish.

The coefficient of x^r in this sum is given by

$$\sum_{j=0}^{r}(-1)^j\binom{n-(m-1)j}{j}\frac{n}{n-(m-1)j}\binom{n-mj}{r-j}$$

for all r such that $r \leq \left\lfloor \frac{n}{m} \right\rfloor$.

Now

$$\sum_{j=0}^{r}(-1)^j\binom{n-(m-1)j}{j}\frac{n}{n-(m-1)j}\binom{n-mj}{r-j} =$$
$$= \frac{n}{r!}\sum_{j=0}^{r}(-1)^j\binom{r}{j}(n-(m-1)j-1)\cdots(n-(m-1)j-r+1) =$$
$$= \Delta^r p(j) = 0$$

for some polynomial with deg $p(x) < r$ and the lemma is proven.

Remark. It is easily verified that for $n > 0$

$$v(n,m,x,s) = \sum_{j=0}^{\frac{n}{m}}(-1)^j\binom{n-(m-1)j}{j}\frac{n}{n-(m-1)j}(x+1)^{n-mj}x^j$$

coincides with the sequence defined by the recurrence

$$v(n+m,m,x,s) = xv(n+m-1,m,x,s) + sv(n,m,x,s)$$

with initial values $v(0,m,x,s) = m$, $v(i,m,x,s) = x^i$, $0 < i < m$.

Lemma 8.3. *Let* $w(n,m,x,s) = (-1)^m t(n,m,x,(-1)^{m-1}s)$ *with*

$$t(n,m,x,s) = \sum\binom{n-j}{(m-1)n-mj}\frac{n}{n-j}x^{(m-1)n-mj}s^j.$$

Then $\sum_{j\geq\frac{(m-2)n}{m-1}} a_{n,m,j}x^{(-mj)\bmod n}s^j = w(n,m,x,s)$ *for* $m > 2$.

Proof: We know already that $a_{n,m,j} \neq 0$ for these values of j only if

$$\frac{(m-2)n}{m-1} \leq j \leq \frac{(m-1)n}{m} \text{ or } \frac{n}{m} \leq n-j \leq \frac{n}{m-1}.$$

Let $k = n - j$.

For $m > 2$ it is evident that if $0 < k_1 + k_2 \leq \frac{n}{m-1}$ for positive numbers k_i there is at most one $k_i \geq \frac{n}{m}$. For $k < \frac{n}{m}$ we have $\left\{\frac{km}{n}\right\} = \frac{km}{n} > \frac{k}{n}$ and therefore $p_k = 0$ by Lemma 7.3.

Thus $\sum_{i=1}^{k-1} b_{n,m,n-i}p_{k-i} = 0$ for each $k < \left\lfloor \frac{n}{m-1} \right\rfloor$.

Newton's formula now tells us that $p_k = -kb_{n,m,n-k}$.

By Lemma 7.3 we have $p_k = (-1)^{k-mk+n}\binom{k}{mk-n}$. This implies $b_{n,m,j} = (-1)^{(m-1)(n-j)+n+1}\binom{n-j}{(m-1)n-mj}\frac{n}{n-j}$ as asserted.

Remark:
It is easy to verify that $t(n,m,x,s)$ may be characterized for $n > 0$ by the recursion

$$t(n,m,x,s) = xs^{m-2}t(n-m+1,m,x,s) + s^{m-1}t(n-m,m,x,s)$$

with the initial values

$$t(0,m,x,s) = m, t(1,m,x,s) = \cdots t(m-2,m,x,s) = 0,$$
$$t(m-1,m,x,s) = (m-1)s^{m-2}$$

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